Application: Linear differential equations

Example 103. (warmup) Solve the **differential equation** (DE) $y' = 2$.

Solution. From calculus, we know that the solutions are of the form $y(t) = 2t + C$.

Comment. To get a unique solution, we need to specify additional information, like an initial condition.

Example 104. (warmup) Solve the **initial value problem** (IVP) $y' = 2$, $y(0) = 1$.

Solution. This has the unique solution $y(t) = 2t + 1$.

Example 105. Which functions $y(t)$ satisfy the differential equation $y' = y$?

Solution. $y(t) = e^t$ and, more generally, $y(t) = Ce^t$. (And nothing else.)

(exponential function) e^t is the unique solution to $y' = y$, $y(0) = 1$.
From here, it follows that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$.

The latter is the Taylor series for e^t at $t = 0$ that we have seen in Calculus II.

Important note. We can actually construct this infinite sum directly from $y' = y$ and $y(0) = 1$.
Indeed, observe how each term, when differentiated, produces the term before it. For instance, $\frac{\mathrm{d}}{\mathrm{d}t} \frac{t^3}{3!} = \frac{t^2}{2!}$.

Example 106. Show that the differential equation $y' = 3y$ is solved by $y(t) = Ce^{3t}$.

Solution. Indeed, if $y(t) = Ce^{3t}$, then $y'(t) = 3Ce^{3t} = 3y(t)$.

Comment. It is important to realize that we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

Example 107. Solve the differential equation $y' = ay$ with initial condition $y(0) = y_0$.

Solution. As in the previous example, the general solution to $y' = ay$ is $y(t) = Ce^{at}$.
Since $y(0) = Ce^0 = C = y_0$, we conclude that the unique solution to the IVP is $y(t) = e^{at} y_0$.

Comment. It looks silly to write $e^{at} y_0$ instead of $y_0 e^{at}$ here, but we will soon replace the number a with a matrix A, and in that case only $e^{At} y_0$ makes sense.

Example 108. Our goal is to solve (systems of) differential equations like:

$$\begin{array}{rcrrr} y_1' &=& 2y_1 & & \\ y_2' &=& -y_1 & +3y_2 & +y_3 \\ y_3' &=& -y_1 & +y_2 & +3y_3 \end{array} \qquad \begin{array}{rcl} y_1(0) &=& 1 \\ y_2(0) &=& 0 \\ y_3(0) &=& 2 \end{array}$$

In matrix form, this becomes

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

The key idea will be to solve $\boldsymbol{y}' = A\boldsymbol{y}$ by introducing e^{At}.

Theorem 109. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At} \boldsymbol{y}_0$.

Recall from Example 107 that the solution to $y' = ay$, $y(0) = y_0$ is $y(t) = e^{at} y_0$. Here, however, At is a matrix and so we need to make sense of the matrix exponential. Next time, we will define e^A by the familiar Taylor series for e^x.

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