Example 101. Let A be the matrix for orthogonally projecting onto $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

Comment. This gives us yet another way to compute projection matrices: we can directly write down the matrices P, D for the diagonalization $A = PDP^T$. The main point here is that the diagonalization of a A nicely reveals all the information about the projection.

(b) Is A invertible, orthogonal, symmetric?

Solution.

(a) The eigenvalues of A are $1, 1, 0$.
The 1-eigenspace of A is W (2-dimensional), and the 0-eigenspace is $W^\perp$ (1-dimensional).
[Make sure this makes perfect sense!]
In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

First, we need to compute a basis for $W^\perp$. After a little work (do it!!), we find $W^\perp = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

Comment. If we choose $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$, we only get $A = PDP^{-1}$.

(b) A is not invertible (because 0 is an eigenvalue) and therefore also cannot be orthogonal.
A is indeed symmetric. That's because $A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$.

By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.

Example 102. Let A be the matrix for **reflecting** through the plane $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) Diagonalize A (without first computing A) as $A = PDP^T$.

(b) Is A invertible, orthogonal, symmetric?

Solution.

(a) This time, the eigenvalues of A are $1, 1, -1$.
The 1-eigenspace of A is W (the plane), and the -1-eigenspace is $W^\perp$ (the normal of the plane).
In order to achieve a diagonalization PDP^T we need to choose P to be orthogonal (which we can do here because the eigenspaces are orthogonal).

As in the previous example, $W^\perp = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

We therefore choose $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$ and, after normalizing columns, $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) A is invertible (because 0 is not an eigenvalue).
By the same reasoning as in the previous example, A is symmetric.
Finally, note that $A^2 = I$ (reflecting twice isn't doing anything), so that $A^{-1} = A$. It follows that A is orthogonal, because $A^{-1} = A = A^T$.

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By the way. Multiplying out $A = PDP^T$, we can find that $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. Similarly, a $n \times n$ matrix corresponds to a reflection (through a hyperplane) if and only if it has a $(n-1)$-dimensional 1-eigenspace and a 1-dimensional -1-eigenspace and these two spaces are orthogonal.

An alternative way of computing reflection matrices. Realize that, if $\boldsymbol{n}$ is the vector orthogonal to the plane (i.e. $\boldsymbol{n}$ is the normal vector of the plane), then reflecting $\boldsymbol{v}$ means sending it to $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n})$.

We already observed that $\boldsymbol{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Hence, the reflection of $\boldsymbol{v}$ is $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n}) = \boldsymbol{v} - 2\boldsymbol{n}\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{\boldsymbol{n}\cdot\boldsymbol{n}} = \boldsymbol{v} - 2\frac{\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{v}}{\boldsymbol{n}^T\boldsymbol{n}} = \left(I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}}\right)\boldsymbol{v}$.

Accordingly, the reflection matrix is $A = I - 2\frac{\boldsymbol{n}\boldsymbol{n}^T}{\boldsymbol{n}^T\boldsymbol{n}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$.

Comment. In other words, we got A from subtracting 2 times the projection matrix onto $\boldsymbol{n}$ from I.