Application: Fibonacci numbers

The numbers $0,1,1,2,3,5,8,13,21,34,\ldots$ are called **Fibonacci numbers**.

They are defined by the recursion $F_{n+1}=F_n+F_{n-1}$ and $F_0=0$, $F_1=1$.

How fast are they growing?

Have a look at ratios of Fibonacci numbers: $\frac{2}{1}=2$, $\frac{3}{2}=1.5$, $\frac{5}{3}=1.6$, $\frac{13}{8}=1.625$, $\frac{21}{13}=1.615$, $\frac{34}{21}=1.619$, ...

These ratios approach the **golden ratio** $\varphi=\frac{1+\sqrt{5}}{2}=1.618\ldots$

In other words, it appears that $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2}$. This indeed follows from Theorem 93 below.

The crucial insight is the following simple observation:

$$F_{n+1}=F_n+F_{n-1} \quad \text{is equivalent to} \quad \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}.$$

In particular, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n\begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

Comment. Recurrence equations are discrete analogs of differential equations. We will later see the same idea applied when we reduce the order of a differential equation by introducing additional equations.

Everything we observe here about Fibonacci numbers holds for other sequences that satisfy similar recursion equations.

Theorem 93. (Binet's formula) $F_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Proof. $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n\begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$

- We already observed that thee recurrence $F_{n+1}=F_n+F_{n-1}$ translates into $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ and, thus, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n\begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

- We therefore diagonalize $T=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as $T=PDP^{-1}$ with

$$D=\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad P=\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}, \quad \lambda_1=\frac{1+\sqrt{5}}{2}\approx 1.618, \quad \lambda_2=\frac{1-\sqrt{5}}{2}\approx -0.618.$$

  Comment. λ_1 is the golden ratio!

- It follows that:

$$\begin{aligned}\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}=T^n\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} &= PD^nP^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix}\frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^{n+1}-\lambda_2^{n+1} \\ \lambda_1^n-\lambda_2^n \end{bmatrix}\end{aligned}$$

- Hence, $F_n=\frac{1}{\sqrt{5}}(\lambda_1^n-\lambda_2^n)$, which is the claimed formula.

□

Comment. For large n, $F_n \approx \frac{1}{\sqrt{5}} \lambda_1^n$ (because λ_2^n becomes very small). In fact, $F_n = \text{round}\left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \right)$.

Back to the quotient of Fibonacci numbers. In particular, because λ_1^n dominates λ_2^n, it is now transparent that the ratios $\frac{F_{n+1}}{F_n}$ approach $\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$. To be precise, note that

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{n+1} - \lambda_2^{n+1})}{\frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} = \frac{\lambda_1 - \lambda_2 \left(\frac{\lambda_2}{\lambda_1} \right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1} \right)^n} \overset{n \to \infty}{\longrightarrow} \frac{\lambda_1 - 0}{1 - 0} = \lambda_1.$$

In fact, it follows from $\lambda_2 < 0$ that the ratios $\frac{F_{n+1}}{F_n}$ approach λ_1 in the alternating fashion that we observed numerically earlier. Can you see that?

Note that, given any Fibonacci-like recursion, we can apply our linear algebra skills in the same fashion. The next example illustrates how this is set up.

Example 94. Suppose the sequence a_n satisfies $a_{n+3} = 3a_{n+2} - 2a_{n+1} + 7a_n$. Write down a matrix-vector version of this recursion.

Solution. $\begin{bmatrix} a_{n+3} \\ a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -2 & 7 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix}$