Review: More on diagonalization

Example 75. (review) In Example 13, we diagonalized $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ as $A = PDP^{-1}$.

We found that one choice for P and D is $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Spell out what that tells us about A!

Solution. The diagonal entries $5, 2, 2$ of D are the eigenvalues of A.
The columns of P are corresponding eigenvectors of A.

- $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ is a 5-eigenvector of A (that is, $A\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$).
- The 2-eigenspace of A is 2-dimensional. A basis is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Lemma 76. A matrix A is diagonalizable if and only if, for every eigenvalue λ that is k times repeated, the λ-eigenspace of A has dimension k.

In short, an $n \times n$ matrix A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A (i.e. "there are enough eigenvectors").

The next two examples illustrate that not all matrices are diagonalizable and that, even if a real matrix is diagonalizable, the eigenvalues and eigenvectors might be complex.

Example 77. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$? Is A diagonalizable?

Solution. The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) = \lambda^2$, which has $\lambda = 0$ as a double root.

However, the 0-eigenspace $\operatorname{null}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is only 1-dimensional.

As a consequence, A is not diagonalizable.

Example 78. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$? Is A diagonalizable?

Solution. The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$.
Hence, the eigenvalues are $\pm i$.

The i-eigenspace $\operatorname{null}\left(\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\right)$ has basis $\begin{bmatrix} i \\ 1 \end{bmatrix}$.

The $-i$-eigenspace $\operatorname{null}\left(\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\right)$ has basis $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Thus, A has the diagonalization $A = PDP^{-1}$ with $D = \begin{bmatrix} i & \\ & -i \end{bmatrix}$ and $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$.

The spectral theorem

Recall that a matrix A is symmetric if and only if $A^T = A$.

Theorem 79. (spectral theorem, long version) Suppose A is a symmetric matrix.

- A is always diagonalizable.
- All eigenvalues of A are real.
- The eigenspaces of A are orthogonal.

Comment. The eigenspaces of A being orthogonal means that eigenvectors for different eigenvalues are always orthogonal.

Important consequence. In the diagonalization $A = PDP^{-1}$, we can choose P to be orthogonal (in which case $P^{-1} = P^T$). In that case, the diagonalization takes the special form $A = PDP^T$, where P is orthogonal and D is diagonal.

Example 80. (review) If A is a 2×2 matrix with $\det(A) = -8$ and eigenvalue 4. What is the second eigenvalue?

Solution. Recall that $\det(A)$ is the product of the eigenvalues (see below). Hence, the second eigenvalue is -2.

$\det(A)$ is the product of the eigenvalues of A.

Why? Recall how we determine the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A. We compute the characteristic polynomial $\det(A - \lambda I)$ and determine the λ_i as the roots of that polynomial.

That means that we have the factorization $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_n - \lambda)\cdots(\lambda_n - \lambda)$. Now, set $\lambda = 0$ to conclude that $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$.

Example 81.

(a) Determine the eigenspaces of the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(b) Diagonalize A as $A = PDP^T$.

Solution.

(a) The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda - 4)(\lambda + 2)$, and so A has eigenvalues $4, -2$.

The 4-eigenspace is $\operatorname{null}\left(\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The -2-eigenspace is $\operatorname{null}\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Important observation. The 4-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the -2-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are indeed orthogonal!

Review. The product of all eigenvalues $-2 \cdot 4 = -8$ equals the determinant $\det(A) = 1 - 9 = -8$.

(b) Note that a usual diagonalization is of the form $A = PDP^{-1}$.

We need to choose P so that $P^{-1} = P^T$, which means that P must be **orthogonal** (meaning orthonormal columns). [Choosing such a P is only possible if the eigenspaces of A are orthogonal.]

Hence, we normalize the two eigenvectors to $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

With $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$, we then have $A = PDP^T$.