Preparing for the Final

MATH 316 — Linear Algebra II
Monday, April 29

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms, as well as the problems below.

(b) Retake both quizzes.

(c) Finally, retake both midterm exams.

Problem 2. Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

(a) Determine the SVD of A.

(b) Determine the best rank 1 approximation of A.

(c) Determine the pseudoinverse of A.

(d) Find the smallest solution to $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(Then, as a mild check, compare its norm to the obvious solution $\boldsymbol{x} = [\begin{array}{ccc} 1 & 1 & 0 \end{array}]^T$.)

Solution.

(a) $A^T A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ has characteristic polynomial

$$\begin{aligned} \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}\right) &= 0 - 1 \cdot \det\left(\begin{bmatrix} 2-\lambda & 0 \\ 1 & 1 \end{bmatrix}\right) + (2-\lambda)\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) \\ &= -(2-\lambda) + (2-\lambda)\underbrace{((2-\lambda)(1-\lambda) - 1)}_{= \lambda^2 - 3\lambda + 1} \\ &= (2-\lambda)(\lambda^2 - 3\lambda) = (2-\lambda)\lambda(\lambda - 3). \end{aligned}$$

Hence, the eigenvalues are $0, 2, 3$.

- The 0-eigenspace $\operatorname{null}\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

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- The 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
- The 3-eigenspace $\operatorname{null}\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Normalized: $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Therefore, $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$.

Next, $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Hence, $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

(b) From the SVD we just computed it follows that the best rank 1 approximation of A is (that is, we keep 1 singular value only) is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} \sqrt{3} \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) The pseudoinverse of A is

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 0 \\ 1/3 & -1/2 \end{bmatrix}.$$

(d) The smallest solution to $A\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is

$$\boldsymbol{x} = A^+\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/2 \\ 1/3 & 0 \\ 1/3 & -1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 2/3 \\ 1/6 \end{bmatrix}.$$

(For comparison, $\|\boldsymbol{x}\| = \sqrt{11/6} \approx 1.354$ is indeed less than $\|\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T\| = \sqrt{2} \approx 1.414$.) □

Problem 3.

(a) Determine the SVD of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$.

(b) Determine the best rank 1 approximation of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$.

(c) Determine the SVD of $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$.

Solution.

(a) $A^TA = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ has 6-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and 4-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We conclude that $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{6} & \\ & 2 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{12}}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\boldsymbol{u}_2 = \frac{1}{\sigma_2}A\boldsymbol{v}_2 = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\boldsymbol{u}_3$ needs to be chosen so that the matrix U is orthogonal. To find such a vector, we can start with a random vector like $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and then apply a step of Gram–Schmidt to produce a vector that is orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$ We normalize this to $\frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

Hence, $A = U\Sigma V^T$ with $U = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{6} & \\ & 2 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(b) From the SVD we just computed it follows that the best rank 1 approximation of A is (that is, we keep 1 singular value only) is

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} \sqrt{6} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(c) $A^TA = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$ has 10-eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and 0-eigenvector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(Hint: We can immediately read off the 0-eigenvector (make sure that's obvious!). It then follows from the spectral theorem that the vector orthogonal to it must be another eigenvector.)

Normalizing, we conclude that $V = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$.

$\boldsymbol{u}_1 = \frac{1}{\sigma_1}A\boldsymbol{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{50}}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We cannot obtain $\boldsymbol{u}_2$ in the same way because $\sigma_2 = 0$. Since for every vector $\boldsymbol{u}_2$, $A\boldsymbol{v}_2 = \sigma_2\boldsymbol{u}_2$, we can choose $\boldsymbol{u}_2$ as we wish, as long as the columns of U are orthonormal in the end.

Let's choose $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (the only other choice is $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$). Then, $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

In summary, $A = U\Sigma V^T$ with $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. □

Problem 4. Find the best approximation of $f(x) = x$ on the interval $[0, 4]$ using a function of the form $y = a + b\sqrt{x}$.

Solution. The best approximation we are looking for is the orthogonal projection of $f(x)$ onto span$\{1, \sqrt{x}\}$, where the dot product of functions is

$$\langle f, g\rangle = \int_0^4 f(t)g(t)\mathrm{d}t.$$

To find an orthogonal basis for span$\{1, \sqrt{x}\}$, following Gram–Schmidt, we compute

$$\sqrt{x} - \begin{pmatrix}\text{projection of}\\ \sqrt{x} \text{ onto } 1\end{pmatrix} = \sqrt{x} - \frac{\langle\sqrt{x}, 1\rangle}{\langle 1, 1\rangle}1 = \sqrt{x} - \frac{4}{3}.$$

In the last step, we used that

$$\langle 1, 1\rangle = \int_0^4 1\mathrm{d}t = 4, \quad \langle\sqrt{x}, 1\rangle = \int_0^4 \sqrt{t}\mathrm{d}t = \left[\frac{1}{3/2}t^{3/2}\right]_0^4 = \frac{16}{3}.$$

Hence, $1, \sqrt{x} - \frac{4}{3}$ is an orthogonal basis for span$\{1, \sqrt{x}\}$.

The orthogonal projection of $f: [0, 4] \to \mathbb{R}$ onto span$\{1, \sqrt{x}\} =$ span$\left\{1, \sqrt{x} - \frac{4}{3}\right\}$ therefore is

$$\frac{\langle f, 1\rangle}{\langle 1, 1\rangle}1 + \frac{\left\langle f, \sqrt{x} - \frac{4}{3}\right\rangle}{\left\langle\sqrt{x} - \frac{4}{3}, \sqrt{x} - \frac{4}{3}\right\rangle}\left(\sqrt{x} - \frac{4}{3}\right) = \frac{1}{4}\int_0^4 f(t)\mathrm{d}t + \frac{9}{8}\left(\sqrt{x} - \frac{4}{3}\right)\int_0^4 f(t)\left(\sqrt{t} - \frac{4}{3}\right)\mathrm{d}t.$$

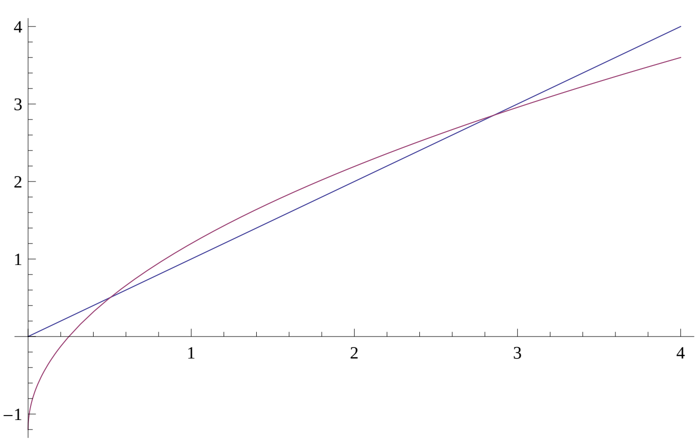
Here, we used that

$$\left\langle\sqrt{x} - \frac{4}{3}, \sqrt{x} - \frac{4}{3}\right\rangle = \int_0^4 \left(\sqrt{t} - \frac{4}{3}\right)^2\mathrm{d}t = \int_0^4 \left(t - \frac{8}{3}\sqrt{t} + \frac{16}{9}\right)\mathrm{d}t = \left[\frac{t^2}{2} - \frac{16}{9}t^{3/2} + \frac{16}{9}t\right]_0^4 = 8 - \frac{128}{9} + \frac{64}{9} = \frac{8}{9}.$$

In our case, this best approximation is

$$\begin{aligned}&\frac{1}{4}\int_0^4 t\mathrm{d}t + \frac{9}{8}\left(\sqrt{x} - \frac{4}{3}\right)\int_0^4 t\left(\sqrt{t} - \frac{4}{3}\right)\mathrm{d}t\\ =\ &\frac{1}{4}\left[\frac{t^2}{2}\right]_0^4 + \frac{9}{8}\left(\sqrt{x} - \frac{4}{3}\right)\left[\frac{2}{5}t^{5/2} - \frac{2}{3}t^2\right]_0^4 = 2 + \frac{12}{5}\left(\sqrt{x} - \frac{4}{3}\right) = \frac{12}{5}\sqrt{x} - \frac{6}{5}.\end{aligned}$$

The plot below confirms the quality of this approximation:



Comment. Computations will be more pleasant on the exam. Do make sure though that the ideas are clear. □

Problem 5. True or false? (As usual, "true" means that the statement is always true.) Explain!

(a) The product of two orthogonal matrices is orthogonal.

(b) $A^T A$ is symmetric for any matrix A.

(c) $A A^T$ is symmetric for any matrix A.

(d) A real $n \times n$ matrix A has real eigenvalues.

(e) The determinant of A is equal to the product of the singular values of A.

(f) The determinant of A is equal to the product of the eigenvalues of A.

(g) If the matrix A is symmetric, then $\langle A\boldsymbol{v}, \boldsymbol{w}\rangle = \langle \boldsymbol{v}, A\boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(h) If the matrix A is orthogonal, then $\langle A\boldsymbol{v}, A\boldsymbol{w}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(i) If $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of A with different eigenvalues, then $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \boldsymbol{0}$.

(j) A is invertible if and only if the only solution to $A\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = \boldsymbol{0}$.

(k) An $n \times n$ matrix A has eigenvalue 0 if and only if it has singular value 0.

(l) An $n \times n$ matrix A has eigenvalue 1 if and only if it has singular value 1.

(m) An $n \times n$ matrix A is singular if and only if 0 is an eigenvalue of A.

(n) An $n \times n$ matrix A is singular if and only if 0 is a singular value of A.

(o) Every symmetric real $n \times n$ matrix A is diagonalizable.

(p) Every symmetric real $n \times n$ matrix A is invertible.

(q) A^T has the same eigenvalues as A.

(r) A^T has the same eigenspaces as A.

(s) A^T has the same characteristic polynomial as A.

(t) Every reflection matrix is invertible.

Solution.

(a) True.

If $A^T = A^{-1}$ and $B^T = B^{-1}$, then $(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}$.

(b) True.

(c) True.

(d) False, because this is not true for all matrices. (Take, for instance, $A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.)

However, by the spectral theorem, a symmetric real $n \times n$ matrix A must have real eigenvalues.

(e) False, but almost true.

Recall that the singular values are all nonnegative, whereas the determinant can be negative.

On the other hand, the absolute value of the determinant of A equals the absolute value of the product of the singular values of A. (Both U and V in $A=U\Sigma V^T$ have determinant ± 1 because they are orthogonal.)

(f) True.

(g) True.

Actually, a matrix A is symmetric if and only if $\langle A\boldsymbol{v}, \boldsymbol{w}\rangle = \langle \boldsymbol{v}, A\boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(h) True.

Actually, a matrix A is orthogonal if and only if $\langle A\boldsymbol{v}, A\boldsymbol{w}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle$ for all vectors $\boldsymbol{v}, \boldsymbol{w}$.

(i) False, because this is not true for all matrices.

However, the statement is true for symmetric matrices by the spectral theorem.

(j) True.

(k) True. Both statements are equivalent to A not being invertible.

(l) False.

(m) True.

(n) True.

(o) True. (That's part of the spectral theorem.)

(p) False.

(q) True. (See Example 89 for this and the next two problems.)

(r) False.

(s) True.

(t) True. (In fact, if A is a reflection matrix, then $A^2 = I$, so that $A^{-1} = A$.) □

Problem 6.

(a) If A has λ-eigenvalue $\boldsymbol{v}$, then A^3 has ______.

(b) A is singular if and only if $\dim \operatorname{null}(A)$ ______.

(c) If $A = \begin{bmatrix} i & 1+2i \\ 3 & 4 \\ 5i & 6-i \end{bmatrix}$, then its conjugate transpose is $A^* =$ ______.

(d) The norm of the vector $\boldsymbol{v} = \begin{bmatrix} 1-i \\ 2i \end{bmatrix}$ is $\|\boldsymbol{v}\| =$ ______.

(e) By Euler's identity, $e^{ix} =$ ______.

(f) What exactly does it mean for a matrix A to have full column rank?

(g) The pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \end{bmatrix}$ is $A^+ =$ ______.

(h) If A is invertible then its pseudoinverse is $A^+ =$ ______.

(i) If A has full column rank then its pseudoinverse is $A^+ =$ ______.

(j) Suppose the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has infinitely many solutions $\boldsymbol{x}$.

Which of these solutions is produced by $A^+\boldsymbol{b}$? ______

(k) Write down the 2×2 rotation matrix by angle θ.

Solution.

(a) If A has λ-eigenvalue $\boldsymbol{v}$, then A^3 has λ^3-eigenvalue $\boldsymbol{v}$.

(b) A is singular (i.e. not invertible) if and only if $\dim \operatorname{null}(A) > 0$.

(c) If $A = \begin{bmatrix} i & 1+2i \\ 3 & 4 \\ 5i & 6-i \end{bmatrix}$, then its conjugate transpose is $A^* = \begin{bmatrix} -i & 3 & -5i \\ 1-2i & 4 & 6+i \end{bmatrix}$.

(d) The norm of the vector $\boldsymbol{v} = \begin{bmatrix} 1-i \\ 2i \end{bmatrix}$ is $\|\boldsymbol{v}\| = \sqrt{|1-i|^2 + 2^2} = \sqrt{6}$.

(e) By Euler's identity, $e^{ix} = \cos(x) + i\sin(x)$.

(f) A matrix A has full column rank if its rank equals the number of columns.

(g) The pseudoinverse of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \end{bmatrix}$ is $A^+ = \begin{bmatrix} 1/3 & 0 \\ 0 & -1/7 \\ 0 & 0 \end{bmatrix}$.

(h) If A is invertible, then $A^+ = A^{-1}$.

(i) If A has full column rank then its pseudoinverse is $A^+ = (A^T A)^{-1} A^T$.

(j) The one of smallest norm.

(k) The 2×2 rotation matrix by angle θ is $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. □