Example 172. Give a basis for the space of all polynomials of degree 3 or less.

Solution. The polynomials $1, x, x^2, x^3$ form a basis for that space.

To see why, recall that the basis vectors need to do two things: they need to span the space and they need to be independent. Equivalently, every element in the space needs to representable as a linear combination of the basis elements (they span) and this representation must be unique (they are independent).

We are familiar with the fact that every polynomial $p(x)$ of degree 3 or less can be written as $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, and that the coefficients a_0, a_1, a_2, a_3 are unique.

Important observation. As a consequence, the $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ can be expressed as $\begin{bmatrix} a_o \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

That is, we can work with the familiar column vectors in any vector space, as soon as we have picked a basis. The next example will illustrate how to translate back and forth.

Example 173. Give a basis for the space of all polynomials $p(x)$ of degree 2 or less such that $p(3) = 0$.

Solution. (calculus) From Calculus, we know that $p(3) = 0$ means that 3 is a root of the polynomial, and that, as a consequence, the polynomial factors as $p(x) = (x-3)q(x)$, where $q(x)$ is another polynomial.

Hence, a basis for our space is $x-3, x(x-3)$.

[That is, we are multiplying $x-3$ with $1, x, x^2 \ldots$ but stop at x because we are restricted to degree 2 or less.]

Solution. (linear algebra) Let us start with the basis $1, x, x^2$ for the space of all polynomials $p(x)$ of degree 2 or less.

Then, we can identify the polynomial $p(x) = a_0 + a_1x + a_2x^2$ with the vector $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

The condition $p(3) = 0$ translates into $a_0 + 3a_1 + 9a_2 = 0$.

In other words, the space of polynomials $p(x)$ of degree 2 or less such that $p(3) = 0$ translates into $\operatorname{null}(\begin{bmatrix} 1 & 3 & 9 \end{bmatrix})$.

A basis for $\operatorname{null}(\begin{bmatrix} 1 & 3 & 9 \end{bmatrix})$ is $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 1 \end{bmatrix}$.

The corresponding polynomials are $-3 + x$ and $-9 + x^2$.

Example 174. Give a basis for the space of all polynomials $p(x)$ of degree 3 or less such that $p(1) = 0$ and $p'(1) = 0$.

Solution. Let us start with the basis $1, x, x^2, x^3$ for the space of all polynomials $p(x)$ of degree 3 or less.

Then, we can identify the polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ with the vector $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

The condition $p(1) = 0$ translates into $a_0 + a_1 + a_2 + a_3 = 0$.

Since $p'(x) = a_1 + 2a_2x + 3a_3x^2$, the condition $p'(1) = 0$ translates into $a_1 + 2a_2 + 3a_3 = 0$.

In other words, the space of all polynomials $p(x)$ of degree 3 or less such that $p(1) = 0$ and $p'(1) = 0$ translates into $\operatorname{null}\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}\right)$.

A basis for $\operatorname{null}\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}\right)$ is $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. (Fill in the details!)

The corresponding polynomials are $1 - 2x + x^2$ and $2 - 3x + x^3$.

[Check that they indeed satisfy $p(1) = 0$ and $p'(1) = 0$.]

Comment. Let's note that it was to be expected from the beginning that the space is 2-dimensional. The space of all polynomials $p(x)$ of degree 3 or less has dimension 4. Since we impose 2 (independent) conditions, the dimension of our space is $4 - 2 = 2$.

Armin Straub
straub@southalabama.edu

An inner product on function spaces

On the space of, say, (piecewise) continuous functions $f: [a, b] \to \mathbb{R}$, it is natural to consider the dot product

$$\langle f, g \rangle = \int_a^b f(t) g(t) \mathrm{d}t.$$

Why? A (sensible) dot product provides a (sensible) notion of distance between functions. The dot product above is the continuous analog of the usual dot product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{t=1}^n x_t y_t$ for vectors in $\mathbb{R}^n$. Do you see it?!

As a consequence, once we have the dot product, we can orthogonally project functions onto spaces of simple functions. In other words, we can compute best approximations of functions by simple functions (for instance, best quadratic approximations).

Why continuous? We need that any product $f(x)g(x)$ is integrable. That means we cannot work with all functions. Continuity is certainly sufficient. In fact, the right condition is that $f(x)^2$ should be integrable on $[a, b]$ (i.e. $f(x)$ is square-integrable). Such a function is said to be in $\mathcal{L}^2[a, b]$.

Example 175. What is the orthogonal projection of $f: [a, b] \to \mathbb{R}$ onto the space of constant functions (that is, $\mathrm{span}\{1\}$)?

Solution. The orthogonal projection of $f: [a, b] \to \mathbb{R}$ onto $\mathrm{span}\{1\}$ is

$$\frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} 1 = \frac{\int_a^b f(t) 1 \mathrm{d}t}{\int_a^b 1^2 \mathrm{d}t} = \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t.$$

This is the average of $f(x)$ on $[a, b]$.

Comment. Makes perfect sense, doesn't it? Intuitively, the best approximation of a function by a constant should indeed be the one where the constant is the average.

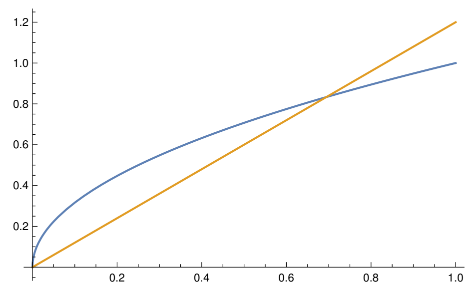
Example 176. Find the best approximation of $f(x) = \sqrt{x}$ on the interval $[0, 1]$ using a function of the form $y = ax$.

Solution. The orthogonal projection of $f\colon [0, 1] \to \mathbb{R}$ onto $\mathrm{span}\{x\}$ is

$$\frac{\langle f, x \rangle}{\langle x, x \rangle} x = \frac{\int_0^1 f(t) t \mathrm{d}t}{\int_0^1 t^2 \mathrm{d}t} x = 3x \int_0^1 t f(t) \mathrm{d}t.$$

In our case, the best approximation is

$$3x \int_0^1 t\sqrt{t} \mathrm{d}t = 3x \int_0^1 t^{3/2} \mathrm{d}t = 3x \left[\frac{1}{5/2} t^{5/2} \right]_0^1 = \frac{6}{5} x.$$



Armin Straub
straub@southalabama.edu