Example 145. We only discuss linear differential equations (DEs). Non-linear DEs include $y' = y^2 + 1$ or the second-order equation $y'' = \sin(ty') + y$.

The order of a DE indicates the highest occuring derivative.

Note, however, that $y'' = \sin(t)y' + y$ is a linear DE, because y and its derivatives occur linearly.

We will see here how to solve those linear DEs which have constant coefficients. That is, the coefficients of y are constants, as opposed to functions (like $\sin(t)$) depending on t.

Review.

- The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

  Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

- If we have the diagonalization $A = PDP^{-1}$, then $e^A = Pe^DP^{-1}$.

- If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then $e^A = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$ and $e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$.

Example 146. Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + y$ becomes $y_2' = 2y_2 + y_1$.

Therefore, $y'' = 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\boldsymbol{y}$.

Comment. This is further indication why we would care about systems of differential equations, even if we work with just one function.

Example 147. Write the (third-order) differential equation $y''' = 3y'' - 2y' + y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix}\boldsymbol{y}$.

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Example 148. (extra) Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution. Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.

- We first diagonalize $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
  - $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$, so the eigenvalues are ± 1.
  - The 1-eigenspace $\operatorname{null}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  - The -1-eigenspace $\operatorname{null}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
  - Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- Compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y} = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t \\ -e^{-t} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{bmatrix} \end{aligned}$$

Check. Indeed, $y_1 = \frac{1}{2}(e^t + e^{-t})$ and $y_2 = \frac{1}{2}(e^t - e^{-t})$ satisfy the system of differential equations $y_1' = y_2$ and $y_2' = y_1$ as well as the initial conditions $y_1(0) = 1$, $y_2(0) = 0$.

Comment. You have actually met these functions in Calculus! $y_1 = \cosh(t)$ and $y_2 = \sinh(t)$. Check out the next example for the connection to $\cos(t)$ and $\sin(t)$.

Example 149. (extra) What is the system of differential equations characterizing $y_1(t) = \cos(t)$ and $y_2 = \sin(t)$?

Solution. Since $y_1' = -\sin(t) = -y_2$ and $y_2' = \cos(t) = y_1$, the system is $\boldsymbol{y}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Comment. If you solve this system as in the previous example, then you find $\boldsymbol{y} = \frac{1}{2}\begin{bmatrix} e^{it} + e^{-it} \\ -ie^{it} + ie^{-it} \end{bmatrix} = \begin{bmatrix} (e^{it} + e^{-it})/2 \\ (e^{it} - e^{-it})/(2i) \end{bmatrix}$. It follows from Euler's formula, that this equals $\boldsymbol{y} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$. Alternatively, you have just discovered and proven Euler's formula using differential equations!

Also, we see from here that $\cos(it) = \cosh(t)$ and $\sin(it) = i\sinh(t)$.