Example 82. (review) In Example 10, we diagonalized $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ as $A = PDP^{-1}$.

We found that one choice for P and D is $P = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Spell out what that tells us about A!

Solution. The diagonal entries $5, 2, 2$ of D are the eigenvalues of A.
The columns of P are corresponding eigenvectors of A.

- $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ is a 5-eigenvector of A (that is, $A\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$).
- The 2-eigenspace of A is 2-dimensional. A basis is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Example 83. A matrix A is diagonalizable if and only if, for every eigenvalue λ that is k times repeated, the λ-eigenspace of A has dimension k.

In short, an $n \times n$ matrix A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A (i.e. "there are enough eigenvectors").

Example 84. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$? Is A diagonalizable?

Solution. The characteristic polynomial is $\det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) = \lambda^2$, which has $\lambda = 0$ as a double root.

However, the 0-eigenspace $\text{null}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is only 1-dimensional.

As a consequence, A is not diagonalizable.

The spectral theorem

Recall that a matrix A is symmetric if and only if $A^T = A$.

Theorem 85. (spectral theorem, long version) Suppose A is symmetric matrix.

- A is always diagonalizable.
- All eigenvalues of A are real.
- The eigenspaces of A are orthogonal.

Comment. The eigenspaces of A being orthogonal means that eigenvectors for different eigenvalues are always orthogonal.

Important consequence. In the diagonalization $A = PDP^{-1}$, we can choose P to be orthogonal (in which case $P^{-1} = P^T$). In that case, the diagonalization takes the special form $A = PDP^T$, where P is orthogonal and D is diagonal. More next time!