1 Preparing for Midterm 2

- These problems are taken from the lectures to help you prepare for our upcoming midterm exam. You can find solutions to all of these in the lecture sketches.
- I will also post additional practice problems before the end of the week.

Example 1. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$?

Theorem 2. (spectral theorem)

Example 3. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. What is the geometric description of this linear map? What are the eigenvalues and eigenvectors?

Example 4. Find the 3×3 matrix A for reflecting about the plane spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ in two different ways:

(a) By writing down the diagonalization of A.

(b) By realizing that, if $\boldsymbol{n}$ is the vector orthogonal orthogonal to the plane, then reflecting $\boldsymbol{v}$ means sending it to $\boldsymbol{v} - 2(\text{projection of } \boldsymbol{v} \text{ onto } \boldsymbol{n})$.

Example 5. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$. Without computations, what are the eigenvalues and eigenvectors?

Example 6. What are the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$?

Example 7. Diagonalize the symmetric matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ as $A = PDP^T$.

Example 8. By the spectral theorem, every symmetric matrix A can be written as $A = VDV^T$ for a diagonal matrix D and an orthogonal matrix V. What about A^{-1}? A^n?

Example 9. If $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, what is D^{-1}?

Example 10. Consider the 3×3 matrix A of a reflection through a plane (containing the origin). What can we say about A?

Example 11. Similar to the last example, let 3×3 be the matrix A of a projection onto a plane (containing the origin). What can we say about A?

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Example 12. $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $J\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$. What is the geometric description of this linear map? What are the eigenvalues and eigenvectors?

Example 13. What is $\frac{1}{2+3i}$?

- In general, $\frac{1}{z} =$ [ ].
- The **absolute value** of the complex number $z = x + iy$ is $|z| =$ [ ].
- The **norm** of the complex vector $\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is $\|\boldsymbol{z}\| =$ [ ].

Definition 14.

- For any matrix A, its **conjugate transpose** is $A^* =$ [ ].
- The **dot product** of complex vectors is $\langle \boldsymbol{v}, \boldsymbol{w} \rangle =$ [ ].
- A complex $n \times n$ matrix A is **unitary** if [ ].

Example 15. What is the norm of the vector $\begin{bmatrix} 1-i \\ 2+3i \end{bmatrix}$?

Example 16. Determine A^* if $A = \begin{bmatrix} 2 & 1-i \\ 3+2i & i \end{bmatrix}$.

Let A be a real matrix. If $\boldsymbol{v}$ is a λ-eigenvector, then $\bar{\boldsymbol{v}}$ is [ ].

Example 17. Find a unitary matrix Q whose first column is a multiple of $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x =$ [ ] and $y =$ [ ].

Every complex number z can be written in **polar form** as $z =$ [ ], with $r = |z|$.

Theorem 18. (Euler's identity) $e^{i\theta} =$ [ ]

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Example 19. (multiplication of complex numbers) Gives a geometric interpretation of what multiplication of complex numbers means.

In particular, what is the geometric interpretation of multiplying with i?

Example 20. (rotation matrices) Write down the 2×2 rotation matrix by angle θ.

> **(Singular value decomposition)**
> Every $m \times n$ matrix A can be decomposed as

Example 21. Determine the SVD of $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

Example 22. Determine the SVD of $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

Example 23.

(a) Determine the SVD of $A = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$.

(b) Determine the SVD of $A = \begin{bmatrix} 1 & 5 \\ -7 & 5 \end{bmatrix}$.

Example 24. Determine the SVD of $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Example 25. (least squares) Recall that if $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent, it is often useful to determine a least squares solution by solving $\boxed{}$.

> The **pseudoinverse** of an $m \times n$ matrix A with SVD $A = U\Sigma V^T$ is
>
> $$A^+ = \boxed{},$$

- If A is invertible, then $A^+ = \boxed{}$.

  Why? A is invertible if and only if Σ is invertible. Clearly, $\Sigma^{-1} = \Sigma^+$.
  Hence, $A^{-1} = (U\Sigma V^T)^{-1} = V\Sigma^{-1}U^T = V\Sigma^+U^T = A^+$.

- If A has full column rank, then $A^+ = \boxed{}$.

- The pseudoinverse of A^+ is $A^{++} = \boxed{}$.

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Example 26. What is the pseudoinverse of $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$?

Example 27. Determine the pseudoinverse of $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ in two ways.

First, using the SVD and, second, using the fact that A has full column rank.

Example 28. What is the pseudoinverse of $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$?

Example 29. What is the pseudoinverse A^+ of $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$? Compute A^+A and AA^+.

Example 30. Find the smallest solution to $A\boldsymbol{x} = [\ 6\ ]$ with $A = [\ 2\ \ 0\ \ 0\ ]$ in two ways: directly (because the equation is so simple) and using the pseudoinverse of $A = [\ 2\ \ 0\ \ 0\ ]$.

Example 31.

(a) Find the pseudoinverse of $A = [\ 1\ \ 2\ \ 3\ ]$.

(b) Find the smallest solution to $x_1 + 2x_2 + 3x_3 = 6$.

Example 32. Find the pseudoinverse of $A = [\ a_1\ \ a_2\ \ a_3\ ]$.

Theorem 33. (matrix approximation lemma) Suppose A is a $m \times n$ matrix, and we want to approximate A using a matrix B of rank s (smaller than the rank of A).

Then, the best such approximation is $B =$ [ ]

Example 34. Solve the differential equation $y' = 2$.

Example 35. Solve the initial value problem $y' = 2$, $y(0) = 1$.

Example 36. Which functions $y(t)$ satisfy the differential equation $y' = y$?

Example 37. Show that the differential equation $y' = 3y$ is solved by $y(t) = Ce^{3t}$.

Example 38. Solve the differential equation $y' = ay$ with initial condition $y(0) = y_0$.

Example 39. Which of the following DEs, if any, is linear?

$$y' = y^2 + 1, \quad y'' = \sin(ty') + y, \quad y'' = \sin(t)y' + y.$$

Example 40. Express the following system of differential equations in matrix form:

$$\begin{array}{rcrrrcrcl} y_1' &=& 2y_1 & & & \qquad & y_1(0) &=& 1 \\ y_2' &=& -y_1 & +3y_2 & +y_3 & & y_2(0) &=& 0 \\ y_3' &=& -y_1 & +y_2 & +3y_3 & & y_3(0) &=& 2 \end{array}$$

Example 41. Diagonalize $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Example 42. Write the (second-order) differential equation $y'' = 2y' + y$ as a system of (first-order) differential equations.

Example 43. Write $y''' = 3y'' - 2y' + y$ in the form $\boldsymbol{y}' = A\boldsymbol{y}$.

Theorem 44. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) =$

Definition 45. Let A be $n \times n$. The matrix exponential is $e^A =$.

Example 46. Suppose $A = PDP^{-1}$. Then, what is A^n?

Theorem 47. Suppose $A = PDP^{-1}$. Then, $e^A =$.

Example 48. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, then $A^{100} =$.

Example 49. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, then $e^A =$.

Example 50. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Example 51. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Example 52. What is the system of differential equations characterizing $y_1(t) = \cos(t)$ and $y_2 = \sin(t)$?

Example 53. Solve the second-order differential equation

$$y'' = y' + 2y, \quad y(0) = 1, \quad y'(0) = 3,$$

by first converting it to a system of first-order differential equations.

Example 54. What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues $3, 3, 3$? What if the matrix is 5×5 and has eigenvalues $4, 4, 3, 3, 3$?