Example 155. (warmup) The spectral theorem says that symmetric (real) $n \times n$ matrices are always diagonalizable, have real eigenvalues, and orthogonal eigenspaces.

Why is it (strictly speaking) incorrect to say that the eigenvectors are orthogonal?

Solution. Think, for instance, of the symmetric matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Its eigenvalues are $0, 0$, and the 0-eigenspace is $\mathbb{R}^2$ (make sure these statements are obvious!).

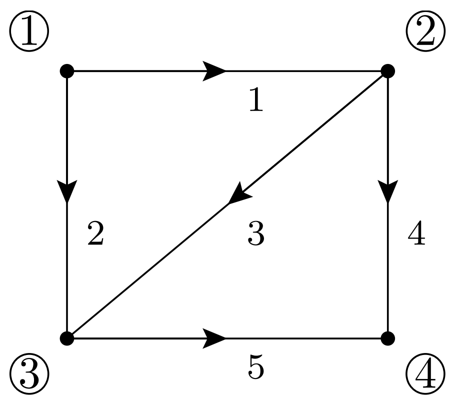
In other words, all vectors are eigenvectors but certainly it is false that all vectors are orthogonal.

(For instance, take $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.)

Correct statements involving the eigenvectors are:

- Eigenvectors with different eigenvalues are orthogonal.
- We can choose an orthogonal basis for $\mathbb{R}^n$ consisting of eigenvectors.

9 Application: directed graphs



(Directed) graphs appear in countless applications, obviously including network and circuit analysis.

- The arrows ("direction of flow") are what makes the graph directed.
- In our discussion, we will allow no edges from a node to itself (no "self-loops"), and at most one edge between nodes (no "multi-edges").

Definition 156. Let G be a graph with m edges and n nodes.

The **edge-node incidence matrix** of G is the $m \times n$ matrix A with

$$A_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j, \\ +1, & \text{if edge } i \text{ enters node } j, \\ 0, & \text{otherwise.} \end{cases}$$

Example 157. Determine the edge-node incidence matrix A of the graph G above.

Solution. $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

Observations. Each column represents a node, and each row represents an edge.

For instance, the first column represents the first node, which has two outgoing edges (namely, #1 and #2) corresponding to the two -1's.

As a consequence, each row (edge) has exactly two entries, one -1 (start node), one $+1$ (end node).

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9.1 Meaning of the (right) null space

The $\boldsymbol{x}$ in $A\boldsymbol{x}$ is assigning values to each node.

You may think of assigning **potentials** to each node. In our running example:

$$A\boldsymbol{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ -x_2 + x_3 \\ -x_2 + x_4 \\ -x_3 + x_4 \end{bmatrix}$$

Crucial observation. $A\boldsymbol{x} = \boldsymbol{0} \iff \boldsymbol{x}$ assigns the same value to nodes connected by an edge

As a consequence, $\boldsymbol{x}$ assigns the same value to nodes that are connected in any way (via a path of edges).

Example. In our running example, $\dim \operatorname{null}(A) = 1$ and $\operatorname{null}(A)$ is spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

This reflects the fact that our graph is **connected**: that is, each node connects to every other node by a sequence of edges.

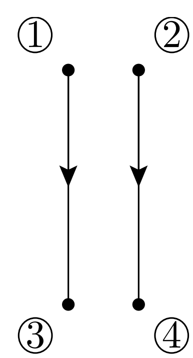
(nullspace of edge-node incidence matrix)

$\dim \operatorname{null}(A)$ is the number of **connected components**.

In particular, the graph is connected if and only if $\dim \operatorname{null}(A) = 1$.

Comment. For large graphs, disconnection is not something you can just see by looking at a graph. But, now, we can always find out by computing $\dim \operatorname{null}(A)$ using Gaussian elimination!

Example 158. (homework) Consider the graph to the right. Determine its edge-node incidence matrix A. Give a basis for $\operatorname{null}(A)$. What do we conclude?



Solution. The edge-node incidence matrix is $A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$.

$\operatorname{null}(A)$ has the basis: $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. (Do it!)

Since $\operatorname{null}(A)$ is not spanned by $[1, 1, 1, 1]^T$, we conclude that the graph is not connected.

From $\dim \operatorname{null}(A) = 2$, we actually know that the graph has 2 connected components.

In fact, from our basis, we can even read off which nodes belong to each connected component (#1, #3 in one, and #2, #4 in the other).