Example 149. Diagonalize, if possible, the matrix $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$.

Solution. The eigenvalues of A are $2, 2$.

However, the 2-eigenspace $\operatorname{null}\left(\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}\right)$ is only 1-dimensional.

Hence, A is not diagonalizable.

Definition 150. A **λ-Jordan block** is a matrix of the form $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$.

Note that if this matrix is $m \times m$, then its only eigenvalue is λ (repeated m times).
As in the previous example, the λ-eigenspace is 1-dimensional (which is as small as possible).

Theorem 151. (Jordan normal form) Every $n \times n$ matrix A can be written as $A = PJP^{-1}$, where J is a block diagonal matrix

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix}$$

with each J_i a Jordan block. J is called the **Jordan normal form** of A.

Up to the ordering of the Jordan blocks, the Jordan normal form of A is unique.

Comment. If A is diagonalizable, then J is just a usual diagonal matrix.

Example 152.

(a) What are the possible Jordan normal forms of a 3×3 matrix with eigenvalues $3, 3, 3$?

(b) What are the possible Jordan normal forms of a 4×4 matrix with eigenvalues $3, 3, 3, 3$?

(c) **(homework)** What if the matrix is 5×5 and has eigenvalues $4, 4, 3, 3, 3$?

Solution.

(a) $\begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix}, \begin{bmatrix} 3 & & \\ & 3 & 1 \\ & & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{bmatrix}$

The dimension of the 3-eigenspace equals the number of Jordan blocks: 3, 2, 1, respectively.

(b) Now, there are 5 possibilities:

$\begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{bmatrix}$

The dimension of the 3-eigenspace equals the number of Jordan blocks: 4, 3, 2, 2, 1, respectively.

(c) $\begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & & & \\ & 3 & 1 & & \\ & & 3 & & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}$

Note that this is just all possible (namely, 3) Jordan normal forms of a 3×3 matrix with eigenvalues $3, 3, 3$ combined with all possible (namely, 2) Jordan normal forms of a 2×2 matrix with eigenvalues $4, 4$. In total, that makes $3 \cdot 2 = 6$ possibilities.

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Example 153. The matrix exponential shares many other properties of the usual exponential:

- $e^A e^B = e^{A+B} = e^B e^A$ if $AB = BA$

  Why the condition $AB = BA$? By the Taylor series, $e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \ldots$ In order to simplify that to

  $$e^A e^B = \left(I + A + \frac{A^2}{2!} + \ldots\right)\left(I + B + \frac{B^2}{2!} + \ldots\right),$$

  we need that $(A+B)^2 = A^2 + AB + BA + B^2$ is the same as $A^2 + 2AB + B^2$. That's only the case if $AB = BA$.

- e^A is invertible and $(e^A)^{-1} = e^{-A}$

  Why? That actually follows from the previous property.

Example 154. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Solution.

- If $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix}$, then the solution is $\boldsymbol{y}(t) = e^{At} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  The only difficulty is in computing e^{At} since we already observed that A is not diagonalizable.

- Write $A = \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} = 2I + N$ with $N = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$. Note that $2I$ and N commute.

  Hence, $e^{At} = e^{2It + Nt} = e^{2It} e^{Nt}$.

- Note that $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $e^{Nt} = I + Nt + \frac{t^2}{2!} N^2 + \ldots = I + Nt = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$.

- Combined, $e^{At} = e^{2It + Nt} = e^{2It} e^{Nt} = \begin{bmatrix} e^{2t} & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ & e^{2t} \end{bmatrix}$.

  In particular, $\boldsymbol{y}(t) = e^{At} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} (t-1)e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} t-1 \\ 1 \end{bmatrix} e^{2t}$.

Check. We should verify that $y_1 = (t-1)e^{2t}$ and $y_2 = e^{2t}$ satisfy $y_1' = 2y_1 + y_2$ and $y_2' = 2y_2$.
Indeed, $y_1' = e^{2t} + (t-1)2e^{2t}$ equals $2y_1 + y_2 = 2(t-1)e^{2t} + e^{2t}$.

Comment. For applications, having solutions like $te^{\lambda t}$ or $t\cos(\lambda t)$ (when the eigenvalues are imaginary) is connected to the phenomenon of **resonance**, which you may have already seen.