Example 107. (rotation matrices) Recall that $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the $90°$ rotation matrix.

So, J is a matrix analog of i. In particular, we have $J^2 = -I$ just like $i^2 = -1$.
More generally, the complex number $x + iy$ can be represented by $xI + yJ = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$.
In particular, $e^{i\theta} = \cos\theta + i\sin\theta$ (rotation by angle θ) corresponds to the **rotation matrix** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
Comment. Every orthogonal 2×2 matrix with determinant 1 is a rotation matrix.

7 Singular value decomposition

(Singular value decomposition)
Every $m \times n$ matrix A can be decomposed as $A = U\Sigma V^T$, where

- Σ is a (rectangular) diagonal matrix with nonnegative entries, $(m \times n)$

The diagonal entries σ_i are called the **singular values** of A.

- U is orthogonal, $(m \times m)$
- V is orthogonal. $(n \times n)$

Comment. If A is symmetric, then the singular value decomposition is already provided by the spectral theorem (the diagonalization of A). Moreover, in that case, $V = U$.
Important observations. If $A = U\Sigma V^T$, then $A^T A = V\Sigma^T \Sigma V^T$.

- Note that $\Sigma^T \Sigma$ is an $n \times n$ diagonal matrix. Its entries are σ_i^2 (the squares of the entries in Σ).
- $A^T A$ is a symmetric matrix! (Why?!) Hence, by the spectral theorem, we are able to find V and $\Sigma^T \Sigma$.

In other words, V is obtained from the (orthonormally chosen) eigenvectors of $A^T A$. Likewise, the entries of $\Sigma^T \Sigma$ are the eigenvalues of $A^T A$; their square roots are the entries of Σ, the singular values.
Finally, the equation $AV = U\Sigma$ allows us to determine U. How?! (Hint: $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$)

This results in the following **recipe** to determine the SVD $A = U\Sigma V^T$ for any matrix A.
Find an orthonormal basis of eigenvectors $\boldsymbol{v}_i$ of $A^T A$. Let λ_i be the eigenvalue of $\boldsymbol{v}_i$.

- V is the matrix with columns $\boldsymbol{v}_i$.
- Σ is the diagonal matrix with entries $\sigma_i = \sqrt{\lambda_i}$.
- U is the matrix with columns $\boldsymbol{u}_i = \frac{1}{\sigma_i} A\boldsymbol{v}_i$. If needed, fill in additional columns to make U orthogonal.

Example 108. Determine the SVD of $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

Solution. $A^T A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ has 2-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and 8-eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Since $A^T A = V\Sigma^2 V^T$ (here, $\Sigma^T \Sigma = \Sigma^2$), we conclude that $V = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{2} & \\ & \sqrt{8} \end{bmatrix}$.
From $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$, we find $\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
Likewise, $\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence, $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Check that, indeed, $A = U\Sigma V^T$!
Comment. For applications, it is common to arrange the singular values in decreasing order.
If we do that, we instead get $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sqrt{8} & \\ & \sqrt{2} \end{bmatrix}$, $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.