

## 1 Review

### 1.1 Matrix calculus

**Example 1.** Matrix multiplication is not commutative!

- $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$
- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix}$

**Example 2.**  $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [14]$  whereas  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ .

If you know about the dot product, do you see a connection with the first case?

**Example 3.** Suppose  $A$  is  $m \times n$  and  $B$  is  $p \times q$ . When does  $AB$  make sense? In that case, what are the dimensions of  $AB$ ?

$AB$  makes sense if  $n = p$ . In that case,  $AB$  is a  $m \times q$  matrix.

**Example 4.**  $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

On the RHS we have the **identity matrix**, usually denoted  $I$  or  $I_2$  (since it's the  $2 \times 2$  identity matrix here).

Hence, the two matrices on the left are inverses of each other:  $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ .

**Example 5.** The following formula immediately gives us the inverse of a  $2 \times 2$  matrix (if it exists). It is worth remembering!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{provided that } ad-bc \neq 0$$

Let's check that!  $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & -cb+ad \end{bmatrix} = I_2$

In particular, a  $2 \times 2$  matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is invertible  $\iff ad-bc \neq 0$ .

Recall that this is the **determinant**:  $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$ .

$$\det(A) = 0 \iff A \text{ is not invertible}$$

## 1.2 Eigenvalues and eigenvectors

If  $Ax = \lambda x$  (and  $x \neq \mathbf{0}$ ), then  $x$  is an **eigenvector** of  $A$  with **eigenvalue**  $\lambda$  (just a number).

Note that for the equation  $Ax = \lambda x$  to make sense,  $A$  needs to be a square matrix (i.e.  $n \times n$ ).

Key observation:

$$\begin{aligned} Ax &= \lambda x \\ \iff Ax - \lambda x &= \mathbf{0} \\ \iff (A - \lambda I)x &= \mathbf{0} \end{aligned}$$

This homogeneous system has a nontrivial solution  $x$  if and only if  $\det(A - \lambda I) = 0$ .

To find eigenvectors and eigenvalues of  $A$ :

(a) First, find the eigenvalues  $\lambda$  by solving  $\det(A - \lambda I) = 0$ .

$\det(A - \lambda I)$  is a polynomial in  $\lambda$ , called the **characteristic polynomial** of  $A$ .

(b) Then, for each eigenvalue  $\lambda$ , find corresponding eigenvectors by solving  $(A - \lambda I)x = \mathbf{0}$ .

More precisely, we find a basis of eigenvectors for the  $\lambda$ -**eigenspace**  $\text{null}(A - \lambda I)$ .

**Example 6.**  $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$  has one eigenvector that is “easy” to see. Do you see it?

**Solution.** Note that  $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ . Hence,  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  is a 2-eigenvector.

**Just for contrast.** Note that  $A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \neq \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . Hence,  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  is not an eigenvector.

**Example 7.**  $\mathbb{R}^3$  is the vector space of all vectors with 3 real entries.

$\mathbb{R}$  itself refers to the set of real numbers. We will later also discuss  $\mathbb{C}$ , the set of complex numbers.

The **standard basis** of  $\mathbb{R}^3$  is  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ .

Recall that vectors form a basis of a space  $V$  if and only iff

- the vectors are independent,
- the vectors span  $V$ .

**Example 8.** Let  $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ . Find the eigenvalues and bases for the eigenspaces of  $A$ .

**Solution.** By expanding by the second column, we find that the characteristic polynomial  $\det(A - \lambda I)$  is

$$\begin{vmatrix} 4-\lambda & 0 & 2 \\ 2 & 2-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(4-\lambda)(3-\lambda) - 2] = (2-\lambda)^2(5-\lambda).$$

Hence, the eigenvalues are  $\lambda = 2$  (with multiplicity 2) and  $\lambda = 5$ .

**Homework.** Determine bases for the corresponding eigenspaces of  $A$ .