Preparing for Midterm #2

MATH 237 — Linear Algebra I
Thursday, Nov 10

Please print your name:

I Computational part

Problem 1. In each case, find a basis for $\operatorname{col}(A), \operatorname{row}(A), \operatorname{null}(A)$.

(a) $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -1 & -2 & -1 & -1 & -3 \\ 2 & 4 & 0 & -6 & 7 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

Problem 2. Find the eigenvalues and bases for the eigenspaces of the following matrices.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 1 & 6 \\ 2 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Problem 3. Suppose there is an epidemic in which, every month, half of those who are well become sick, and a quarter of those who are sick become dead. What is the proportion of dead people in the long term equilibrium.

Problem 4. Consider $H = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

(a) Give a basis for H. What is the dimension of H?

(b) Determine whether the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is in H. What about the vector $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$? If possible, express each vector in terms of your basis for H.

(c) Extend your basis of H to a basis of $\mathbb{R}^3$.

Problem 5. Is it true that $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$?

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II Short answer part

Problem 6. In each case, write down a precise definition or answer.

(a) What is a vector space?

(b) What is the rank of a matrix?

(c) What does it mean for vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ from a vector space to be linearly independent?

(d) What does it mean for vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ to be a basis for a vector space V?

Problem 7. Decide whether the following sets of vectors are a basis of $\mathbb{R}^3$.

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ☐ basis of $\mathbb{R}^3$ ☐ not a basis of $\mathbb{R}^3$

(b) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ ☐ basis of $\mathbb{R}^3$ ☐ not a basis of $\mathbb{R}^3$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ☐ basis of $\mathbb{R}^3$ ☐ not a basis of $\mathbb{R}^3$

(d) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ☐ basis of $\mathbb{R}^3$ ☐ not a basis of $\mathbb{R}^3$

Problem 8. True or false?

(a) Every vector space has a basis.

(b) The zero vector can never be a basis vector.

(c) Every set of linearly independent vectors in V can be extended to a basis of V.

(d) $\operatorname{col}(A)$ and $\operatorname{row}(A)$ always have the same dimension.

(e) If B is the RREF of A, then we always have $\operatorname{col}(A) = \operatorname{col}(B)$.

(f) If B is the RREF of A, then we always have $\operatorname{row}(A) = \operatorname{row}(B)$.

(g) If B is the RREF of A, then we always have $\operatorname{null}(A) = \operatorname{null}(B)$.

(h) If a subspace V of $\mathbb{R}^3$ contains three linearly independent vectors, then always $V = \mathbb{R}^3$.

(i) There are matrices A such that $\operatorname{null}(A)$ is the empty set.

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Problem 9.

(a) What is $\dim \mathrm{null}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \right)$?

(b) If $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$, then $W = \mathrm{row}(A)$ with $A = \ldots$

(c) $\boldsymbol{v}$ is in $\mathrm{null}(A)$ if and only if ...

(d) Let A be a 5×5 matrix with $\dim \mathrm{row}(A) = 5$. What can you say about $\det(A)$?

(e) Let A be a 7×7 matrix with $\dim \mathrm{null}(A) = 1$. What can you say about $\det(A)$?

(f) What are the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 4 \end{bmatrix}$?

(g) Suppose V and W are subspaces of $\mathbb{R}^n$, and that $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a basis for V, and $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ is a basis for W. What can you say about $\dim U$ with $U = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$?

(h) Let A be a 4×3 matrix, whose row space has dimension 2. What is the dimension of $\mathrm{null}(A)$?

(i) Let A be a 3×3 matrix, whose column space has dimension 3. If $\boldsymbol{b}$ is a vector in $\mathbb{R}^3$, what can you say about the number of solutions to the equation $A\boldsymbol{x} = \boldsymbol{b}$?

(j) Let A be a 3×3 matrix, whose column space has dimension 2. What can you say about $\det(A)$?

Problem 10. Suppose that the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ has RREF $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Find a basis for each of $\mathrm{col}(A)$, $\mathrm{row}(A)$ and $\mathrm{null}(A)$.

Problem 11. Let A, B be $n \times n$ matrices such that $AB = \mathbf{0}$. Show that $\det(A) = 0$ or $\det(B) = 0$.

[Recall that it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$.]

Problem 12. If A has eigenvector $\boldsymbol{v}$ with eigenvalue λ, what can you say about eigenvalues and eigenvectors of:

(a) $7A$

(b) A^3

(c) $A - 2I$

Problem 13. Let A be a $m \times n$ matrix.

(a) For each of $\mathrm{col}(A)$, $\mathrm{row}(A)$ and $\mathrm{null}(A)$, state which space $\mathbb{R}^{??}$ they are a subspace of.

(b) Why is $\dim \mathrm{row}(A) + \dim \mathrm{null}(A) = n$?

(c) Suppose that the columns of A are independent. What can you say about the dimensions of $\mathrm{col}(A)$, $\mathrm{row}(A)$ and $\mathrm{null}(A)$?

(d) Suppose that A has rank 2. What can you say about the dimensions of $\mathrm{col}(A)$, $\mathrm{row}(A)$ and $\mathrm{null}(A)$?

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