Review 123.

- **Eigenvector** equation: $A\boldsymbol{x} = \lambda\boldsymbol{x} \iff (A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$

  λ is an **eigenvalue** of $A \iff \underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0.$

- An $n \times n$ matrix A has up to n different eigenvalues λ.

  - The **eigenspace** of λ is $\operatorname{null}(A - \lambda I)$.

    It consists of all eigenvectors of A with eigenvalue λ.
    Since $A - \lambda I$ has determinant 0, $\operatorname{null}(A - \lambda I)$ always has dimension at least 1.

  - If λ has **multiplicity** m (examples below), then the λ-eigenspace has dimension up to m (and at least 1).

    In other words, we find at least 1 eigenvector, and at most m linearly independent eigenvectors of eigenvalue λ.

Comment. "Eigen" is German and means something like "own" or "particular" or "characteristic".

Example 124. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

Solution.

- The **characteristic polynomial** of A is: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & -2 \\ -4 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda) - 8 = \lambda^2 - 2\lambda - 8$

  The **eigenvalues** of A are $\frac{2 \pm \sqrt{4 - 4\cdot(-8)}}{2} = \frac{2 \pm 6}{2}$, so $\lambda_1 = -2$ and $\lambda_2 = 4$.

- For $\lambda_1 = -2$, the eigenspace $\operatorname{null}(A - \lambda_1 I) = \operatorname{null}\left(\begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

  So: $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = -2$.

- For $\lambda_2 = 4$, the eigenspace $\operatorname{null}(A - \lambda_2 I) = \operatorname{null}\left(\begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

  So: $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 4$.

Make sure to quickly check the answer! (One of the vectors we already saw in an example yesterday.)

Important comment. The product of the eigenvalues equals the determinant.

Here, $\lambda_1 \cdot \lambda_2 = -2 \cdot 4 = -8$ and, indeed, $\det(A) = -8$.

To see why this is always the case, note that the characteristic polynomial $\det(A - \lambda I)$ is of the form $(\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$. Setting $\lambda = 0$ gives $\det(A)$ in the former and the product $\lambda_1 \cdots \lambda_n$ of eigenvalues in the latter.

Comment. Note that we can compute the determinant using Gaussian elimination on A. However, we cannot compute the eigenvalues using Gaussian elimination on A. (We can do Gaussian elimination on $A - \lambda I$ but this is much more work because we have to work with polynomials in λ. That's why we will always prefer to use cofactor expansion to compute the characteristic polynomial.)

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Example 125. Find the eigenvalues of $A=\begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ as well as bases for the eigenspaces.

Solution. By expanding by the first row, we find that the characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(3-\lambda)^2-1] = (2-\lambda)(\lambda-2)(\lambda-4).$$

Since $\lambda=2$ is a double root, we say that it has **(algebraic) multiplicity** 2.
Hence, the eigenvalues are $\lambda=2$ (with multiplicity 2) and $\lambda=4$.

- For $\lambda=4$, the eigenspace $\operatorname{null}\left(\begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.
- For $\lambda=2$, the eigenspace $\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Comment. For instance, $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is another basis for the 2-eigenspace (recall that there is infinitely many!). This highlights why we are giving bases for the eigenspaces.

Comment. In these simple cases, we can actually read off the two bases. Try to see that! However, recall that we learned how to compute bases for any null space (see Lecture 14).

E.g., for $\lambda=2$, the RREF of $\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\boldsymbol{x}=\begin{bmatrix} s_1+s_2 \\ s_1 \\ s_2 \end{bmatrix}$ leads to our choice of basis above.

Example 126. Find the eigenvalues of $A=\begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 0 & 3 \end{bmatrix}$ as well as bases for the eigenspaces.

Solution. By expanding by the first row, we find that the characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & 3-\lambda & 1 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)^2.$$

Hence, the eigenvalues are $\lambda=2$ and $\lambda=3$ (with multiplicity 2).

- For $\lambda=2$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.
- For $\lambda=3$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

  This illustrates that an eigenspace can have dimension less than the multiplicity of the eigenvalue.

Comment. Again, try to be able to see these bases. Alternatively, we can always compute:

E.g., for $\lambda=3$, the RREF of $\begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and $\boldsymbol{x}=\begin{bmatrix} 0 \\ s_1 \\ 0 \end{bmatrix}$ leads to our choice of basis above.

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