14 Linear transformations

Throughout, V and W are vector spaces.

Definition 110. A map $T: V \to W$ is a **linear transformation** if

$$T(c\boldsymbol{x} + d\boldsymbol{y}) = cT(\boldsymbol{x}) + dT(\boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \text{ in } V \text{ and all } c, d \text{ in } \mathbb{R}.$$

In other words, a linear transformation respects addition and scaling:

- $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$
- $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$

It necessarily sends the zero vector in V to the zero vector in W:

- $T(\boldsymbol{0}) = \boldsymbol{0}$ [because $T(\boldsymbol{0}) = T(0 \cdot \boldsymbol{0}) = 0 \cdot T(\boldsymbol{0}) = \boldsymbol{0}$]

Example 111. (for context) The **derivative** you know from Calculus I is linear.

Indeed, the map $D: \left\{\begin{array}{c}\text{space of all}\\ \text{differentiable}\\ \text{functions}\end{array}\right\} \to \left\{\begin{array}{c}\text{space of all}\\ \text{functions}\end{array}\right\}$ defined by $f(x) \mapsto f'(x)$ is a linear transformation:

- $\underbrace{D(f(x) + g(x))}_{(f(x)+g(x))'} = \underbrace{D(f(x))}_{f'(x)} + \underbrace{D(g(x))}_{g'(x)}$
- $\underbrace{D(cf(x))}_{(cf(x))'} = \underbrace{cD(f(x))}_{cf'(x)}$

These are among the first properties you learned about the derivate.

Similarly, the **integral** you love from Calculus II is linear:

$$\int_a^b (f(x) + g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x, \quad \int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x$$

In this form, we are looking at a map $T: \left\{\begin{array}{c}\text{space of all}\\ \text{continuous}\\ \text{functions}\end{array}\right\} \to \mathbb{R}$ defined by $T(f(x)) = \int_a^b f(x)\mathrm{d}x$.

Example 112. If A is a $m \times n$ matrix, then $T(\boldsymbol{x}) = A\boldsymbol{x}$ is a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$.

Why? Matrix multiplication is distributive: $A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y}$. (LHS is $T(\boldsymbol{x} + \boldsymbol{y})$ and RHS is $T(\boldsymbol{x}) + T(\boldsymbol{y})$.)
We also know that $A(c\boldsymbol{x}) = cA\boldsymbol{x}$ for scalars c.

Important advanced comments.

- All linear maps $\mathbb{R}^n \to \mathbb{R}^m$ are given by $\boldsymbol{x} \mapsto A\boldsymbol{x}$, for some matrix A.
- The composition of two (compatible) linear maps is another linear map.
- Composition of linear maps corresponds to matrix multiplication!
  If $T(\boldsymbol{x}) = A\boldsymbol{x}$ and $S(\boldsymbol{x}) = B\boldsymbol{x}$, then $(T \circ S)(\boldsymbol{x}) = T(S(\boldsymbol{x})) = A(B\boldsymbol{x}) = (AB)\boldsymbol{x}$.
  This may look simple here, but recall that the matrix product AB takes some work to introduce!

Let's look at some important geometric examples of linear maps $\mathbb{R}^2 \to \mathbb{R}^2$, which are defined by matrix multiplication, that is, by $\boldsymbol{x} \mapsto A\boldsymbol{x}$.

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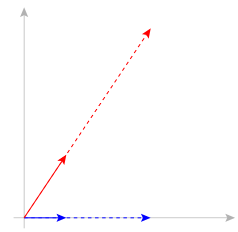
Example 113.

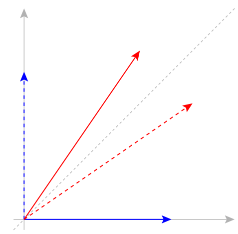


The matrix $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto c\begin{bmatrix} x \\ y \end{bmatrix}$, i.e.

... stretches every vector in $\mathbb{R}^2$ by the same factor c.

Example 114.

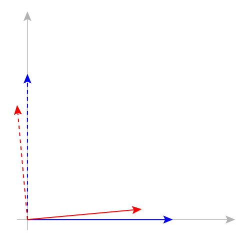


The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$, i.e.

... reflects every vector in $\mathbb{R}^2$ through the line $y = x$.

Example 115.



The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$, i.e.

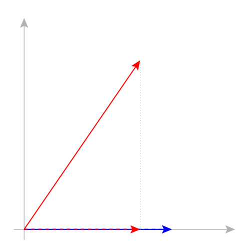
... rotates every vector in $\mathbb{R}^2$ counter-clockwise by $90°$.

Comment. Note that $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, which corresponds to a rotation by $180°$. The point to observe is that the matrix multiplication corresponds precisely to the composition of two rotations by $90°$.

Advanced comment. We have a matrix solution to the equation $X^2 = -I$. The connection to complex numbers is no coincidence! The effect of multiplying a complex number $x + iy$ with i is $i(x + iy) = -y + ix$, the same effect that our matrix A has. Multiplication by i corresponds to a $90°$ rotation of numbers in the complex plane.

[In the same way, the matrices $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ model the complex numbers $a + ib$. Play with it!]

Example 116.



The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$, i.e.

... projects every vector in $\mathbb{R}^2$ onto the x-axis.

15 Eigenvectors and eigenvalues

Throughout, A will be an $n \times n$ matrix.

Definition 117. If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (and $\boldsymbol{x} \neq \boldsymbol{0}$), then $\boldsymbol{x}$ is an **eigenvector** of A with **eigenvalue** λ (just a number).

Example 118. What are the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$?

Solution. Recall that multiplication with A is projection onto the x-axis. Hence:

- $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 1$.
- $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 0$.