Example 101. Let A be a 3×5 matrix. For each of $\operatorname{col}(A)$, $\operatorname{row}(A)$ and $\operatorname{null}(A)$ determine d so that the space is a subspace of $\mathbb{R}^d$.

Solution. $\operatorname{col}(A)$ is a subspace of $\mathbb{R}^3$. $\operatorname{row}(A)$ is a subspace of $\mathbb{R}^5$. $\operatorname{null}(A)$ is a subspace of $\mathbb{R}^5$.

Example 102. Let $A=\begin{bmatrix} 1 & 2 & 0 & 4\\ 2 & 4 & 0 & 8\\ 3 & 6 & 2 & 22\\ 4 & 8 & 0 & 16 \end{bmatrix}$. Find a basis for $\operatorname{col}(A)$, $\operatorname{row}(A)$ and $\operatorname{null}(A)$.

Solution. We compute an echelon form of A:

$$\begin{bmatrix} 1 & 2 & 0 & 4\\ 2 & 4 & 0 & 8\\ 3 & 6 & 2 & 22\\ 4 & 8 & 0 & 16 \end{bmatrix} \overset{\substack{R_2-2R_1\Rightarrow R_2\\ R_3-3R_1\Rightarrow R_3\\ R_4-4R_1\Rightarrow R_4}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 4\\ 0 & 0 & 0 & 0\\ 0 & 0 & 2 & 10\\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_2\Leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 4\\ 0 & 0 & 2 & 10\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can now use Theorem 99 to read off the bases we are interested in:

- The pivot columns are the first and third. Hence, a basis for $\operatorname{col}(A)$ is $\begin{bmatrix} 1\\2\\3\\4 \end{bmatrix}, \begin{bmatrix} 0\\0\\2\\0 \end{bmatrix}$.

  [Make sure you see why it would be horribly wrong to take columns from the echelon form.]

- $\begin{bmatrix} 1\\2\\0\\4 \end{bmatrix}, \begin{bmatrix} 0\\0\\2\\10 \end{bmatrix}$ form a basis for $\operatorname{row}(A)$.

  [Note that it would be horribly wrong to take the first two rows from A.]

- The general solution to $A\boldsymbol{x}=\boldsymbol{0}$ is $\boldsymbol{x}=\begin{bmatrix} -2s_1-4s_2\\ s_1\\ -5s_2\\ s_2 \end{bmatrix} = s_1\begin{bmatrix} -2\\1\\0\\0 \end{bmatrix} + s_2\begin{bmatrix} -4\\0\\-5\\1 \end{bmatrix}$.

  Hence, $\begin{bmatrix} -2\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} -4\\0\\-5\\1 \end{bmatrix}$ are a basis for $\operatorname{null}(A)$.

Let A be $m\times n$, and let r be the **rank** of A, that is, r is the number of pivots.

- $\dim\operatorname{col}(A)=\dim\operatorname{row}(A)=r$
- $\dim\operatorname{null}(A)=n-r$

Example 103. The 4×4 matrix $A=\begin{bmatrix} 1 & 2 & 0 & 4\\ 2 & 4 & 0 & 8\\ 3 & 6 & 2 & 22\\ 4 & 8 & 0 & 16 \end{bmatrix}$ from the previous example has rank 2.

Indeed, $\dim\operatorname{col}(A)=2$, $\dim\operatorname{row}(A)=2$, $\dim\operatorname{null}(A)=4-2=2$ matches our computations.

Example 104. Let A be a 3×5 matrix of rank 2. Determine the dimensions of $\operatorname{col}(A)$, $\operatorname{row}(A)$ and $\operatorname{null}(A)$.

Solution. $\dim\operatorname{col}(A)=2$, $\dim\operatorname{row}(A)=2$, $\dim\operatorname{null}(A)=5-2=3$.

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Example 105. Find a basis for $\operatorname{col}(A), \operatorname{row}(A), \operatorname{null}(A)$ with $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix}$.

Solution. For this simple matrix, we can just "see" the following (make sure you do, too!):

- A basis for $\operatorname{col}(A)$ is: $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ **Why?** Because these two vectors span and are clearly independent.

Note. We would select the same basis, if we computed an echelon form of A and applied Theorem 99(a).

- A basis for $\operatorname{row}(A)$ is: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$ **Why?** Again, because these two vectors span and are clearly independent.

Note. An echelon form of A is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. By Theorem 99(b), an alternative basis for $\operatorname{row}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.
In fact, further computing the RREF, we would select as basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (probably the nicest basis for most purposes).

- A basis for $\operatorname{null}(A)$ is: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Why? We know that $\operatorname{rank}(A) = 2$. Hence, $\dim \operatorname{null}(A) = 3 - 2 = 1$. Therefore, any nonzero vector in $\operatorname{null}(A)$ will be a basis for $\operatorname{null}(A)$. Clearly, $[\ 0\ \ 0\ \ 1\ ]^T$ is one such vector solving $A\boldsymbol{x} = \mathbf{0}$ (why?).

How little we actually know!

Q: *How fast can we solve N linear equations in N unknowns?*

Estimated cost of Gaussian elimination:

$$\begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}$$

- to create the zeros below the first pivot:
  $\Longrightarrow$ on the order of N^2 operations
- if there is N pivots total:
  $\Longrightarrow$ on the order of $N \cdot N^2 = N^3$ operations

- A more careful count places the cost at $\sim \frac{1}{3} N^3$ operations.
- For large N, it is only the N^3 that matters.
  It says that if $N \rightarrow 10N$ then we have to work 1000 times as hard.

That's not optimal! We can do better than Gaussian elimination:

- Strassen algorithm (1969): $N^{\log_2 7} = N^{2.807}$
- Coppersmith–Winograd algorithm (1990): $N^{2.375}$
- ... Stothers–Williams–Le Gall (2014): $N^{2.373}$ (If $N \rightarrow 10N$ then we have to work 229 times as hard.)

Is $N^{2+(\text{a tiny bit})}$ possible? **We don't know!** (People increasingly suspect so.) (Better than N^2 is impossible; why?)

Good news for applications:

- Matrices typically have lots of structure and zeros
  which makes solving so much faster.

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