Recall that a **(vector) space** is a set V of vectors that can be written as a span.

Example 96. The following are important spaces associated with an $m \times n$ matrix A.

- $\operatorname{col}(A)$ (the **column space** of A) is the span of the columns of A.

This is a **subspace** of $\mathbb{R}^m$ (because each column has m entries, and so lives in $\mathbb{R}^m$).

- $\operatorname{row}(A)$ (the **row space** of A) is the span of the rows of A.

This is a **subspace** of $\mathbb{R}^n$ (because each row has n entries, and so lives in $\mathbb{R}^n$).

- $\operatorname{null}(A)$ (the **null space** of A) is the set of all solutions to $A\boldsymbol{x} = \boldsymbol{0}$.

This is also a **subspace** of $\mathbb{R}^n$ (because $\boldsymbol{x}$ has to have n entries for $A\boldsymbol{x}$ to be defined).

12 Bases for null spaces

To find a basis for $\operatorname{null}(A)$:

- find the general solution to $A\boldsymbol{x} = \boldsymbol{0}$, and
- express solutions $\boldsymbol{x}$ as a linear combination of vectors with the free variables as coefficients;
- these vectors form a basis of $\operatorname{null}(A)$.

In particular, the dimension of $\operatorname{null}(A)$ equals the number of free variables.

Example 97. Find a basis for $\operatorname{null}(A)$ with $A = \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 2 & 4 & 5 & 0 & 1 \end{bmatrix}$.

Solution. We eliminate!

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 2 & 4 & 5 & 0 & 1 \end{bmatrix} \overset{R_2 - 2R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix} \overset{R_1 - 2R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 5 & 13 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix}$$

$x_2 = s_1$, $x_4 = s_2$ and $x_5 = s_3$ are our free variables. The solutions to $A\boldsymbol{x} = \boldsymbol{0}$ are:

$$\boldsymbol{x} = \begin{bmatrix} -2s_1 - 5s_2 - 13s_3 \\ s_1 \\ 2s_2 + 5s_3 \\ s_2 \\ s_3 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$.

These vectors are clearly independent (they always are if we proceed as we did!).

If you don't see it, do compute an echelon form! (permute first and third row to the bottom)

Better yet: note that the first vector corresponds to the solution with $s_1 = 1$ and the other free variables $s_2 = 0$, $s_3 = 0$. The second vector corresponds to the solution with $s_2 = 1$ and the other free variables $s_1 = 0$, $s_3 = 0$. The third vector ...

Hence, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $\operatorname{null}(A)$.

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13 Bases for column and row spaces from an echelon form

Example 98. Write down bases for $\operatorname{col}(A)$ and $\operatorname{row}(A)$ with $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Solution. Obviously, $\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is 1-dimensional with basis $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Similarly, $\operatorname{row}(A) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is 1-dimensional with basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

During elimination, we usually do **row operations**. For instance, here, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \overset{R_1 \Leftrightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

- Note that the column spaces changed: $\operatorname{col}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \neq \operatorname{col}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$
- But the row spaces did not change: $\operatorname{row}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \operatorname{row}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$

Row operations preserve the row space but (generally) not the column space.

Theorem 99. Let A be a matrix and B an echelon form of A.

(a) The columns of A (!!) corresponding to the pivot columns of B form a basis for $\operatorname{col}(A)$.

(b) The nonzero rows of B form a basis for $\operatorname{row}(A)$.

(c) In particular, $\dim \operatorname{col}(A) = \dim \operatorname{row}(A)$. They are both equal to the number of pivots.

Why?

(a) If the columns of A are independent, then (obviously!) a basis of $\operatorname{col}(A)$ is given by all columns of A.
Recall that the columns of A are independent
$\Longleftrightarrow$ $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution (namely, $\boldsymbol{x} = 0$),
$\Longleftrightarrow$ A has no free variables.
Hence, a basis for $\operatorname{col}(A)$ is given by the columns of A (!) which do not correspond to a free variable.

(b) As observed above, since we did row operations, we actually have $\operatorname{row}(A) = \operatorname{row}(B)$. The nonzero rows of B are clearly independent and span, and hence are a basis for the row spaces.

Example 100. Find bases for $\operatorname{col}(A)$ and $\operatorname{row}(A)$ if $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$.

Solution. We eliminate!

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 - R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 99(a), a basis for $\operatorname{col}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. $\dim \operatorname{col}(A) = 2$.

[**Warning:** Note that we cannot take the columns of the echelon form to get a basis for $\operatorname{col}(A)$! Can you see why this would definitely lead to an incorrect answer? (Focus on the zeros in the third entry.)]

By Theorem 99(b), a basis for $\operatorname{row}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. $\dim \operatorname{row}(A) = 2$.

[**Warning:** Note that, in general, we cannot just take rows of A to get a basis for $\operatorname{row}(A)$. In this case, the first two rows of A would indeed be a basis for $\operatorname{row}(A)$, but this will not always work. Can you come up with an example?]