11 Vector spaces, bases, dimension

Example 88. (review from exam)

- If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent, then $\boldsymbol{v}_1$ is in $\operatorname{span}\{\boldsymbol{v}_2, \boldsymbol{v}_3\}$.
- If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly dependent, then we cannot say whether $\boldsymbol{v}_1$ is in $\operatorname{span}\{\boldsymbol{v}_2, \boldsymbol{v}_3\}$.

  For instance, the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are dependent, and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in $\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$.

  Similarly, the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ are dependent, but $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is not in $\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right\}$.

Definition 89. A **(vector) space** is a set V of vectors that can be written as a span.
That is, $V = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots\}$ for some bunch of vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$

- Vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$ are called a **basis** of V if
  (a) $V = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots\}$ and
  (b) $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$ are linearly independent.
- The **dimension** of V is the number of elements in such a basis. (It is always the same.)

Example 90. Determine the dimension as well as several bases for the following vector spaces.

(a) $V = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ (b) $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right\}$

Solution.

(a) Note that the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are not a basis of V because they are linearly dependent.

- Since $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, we have $V = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$.

  Thus, because they are also clearly independent, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for V.

  Consequently, $\dim V = 2$.

- By the same reasoning, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis for V.

  In fact, we can shorten the reasoning (more on this later): the two vectors are independent, and we know that $\dim V = 2$, which means it's the right number of vectors.

- Similarly, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for V.

(b) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis for W. Also, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ is a basis for W. But $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ is not a basis for W.
$\dim W = 2$.

Note. Actually, these two are the same vector space: $V = W$.

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Theorem 91. Let $\boldsymbol{x}_p$ be a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$. Then all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ have the form $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_h$ where $\boldsymbol{x}_h$ is a solution to the associated **homogeneous system** $A\boldsymbol{x} = \boldsymbol{0}$.

Example 92. Find the general solution of

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -4 & 2 & 4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

Solution. We eliminate:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ -2 & -4 & 2 & 4 & -2 \end{array}\right] \overset{R_2 + 2R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 0 & 0 & 2 & 2 & 6 \end{array}\right] \overset{\frac{1}{2}R_2 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

Our free variables are $x_2 = s_1$ and $x_4 = s_2$. We read off that $x_1 = 4 - 2s_1 + s_2$, $x_3 = 3 - s_2$. Hence, the general solution is

$$\boldsymbol{x} = \begin{bmatrix} 4 - 2s_1 + s_2 \\ s_1 \\ 3 - s_2 \\ s_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix}}_{\text{particular solution}} + \underbrace{s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{general solution to homogeneous eq.}}.$$

Note. In accordance with Theorem 91, the solution is given as a particular solution $\boldsymbol{x}_p = [\ 4\ \ 0\ \ 3\ \ 0\ ]^T$ plus the general solution to the homogeneous equation $\begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -4 & 2 & 4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Example 93. Find the general solution of

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -4 & 2 & 4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}.$$

Solution. Note that there is one obvious solution: $\boldsymbol{x} = [\ 0\ \ 1\ \ 0\ \ 0\ ]^T$. (Another one is $\boldsymbol{x} = [\ 2\ \ 0\ \ 0\ \ 0\ ]^T$.) Hence, reusing our previous insight, the general solution is

$$\boldsymbol{x} = \begin{bmatrix} 4 - 2s_1 + s_2 \\ s_1 \\ 3 - s_2 \\ s_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular solution}} + \underbrace{s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{general solution to homogeneous eq.}}.$$

Definition 94. $\operatorname{null}(A)$ (the **null space** of A) is the set of all solutions to $A\boldsymbol{x} = \boldsymbol{0}$.

Example 95. Find the dimension and a basis for $\operatorname{null}(A)$ with $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -4 & 2 & 4 \end{bmatrix}$.

Solution. The general solution to $A\boldsymbol{x} = \boldsymbol{0}$ are all vectors $s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. In other words,

$$\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

The two vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ are clearly independent, and so are a basis for $\operatorname{null}(A)$. So, $\dim \operatorname{null}(A) = 2$.