Example 85. Decide whether the following vectors are linearly independent.

(a) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}$ **Solution.** These vectors are linearly dependent.

Why? Because we have the obvious linear dependence relation $2\boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{0}$.

(b) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}$ **Solution.** These vectors are linearly dependent.

Why? Because we have the obvious linear dependence relation $2\boldsymbol{v}_1 + 0\boldsymbol{v}_2 - \boldsymbol{v}_3 = \boldsymbol{0}$.

(c) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ **Solution.** These vectors are linearly dependent.

Moral. More than n vectors in $\mathbb{R}^n$ are always linearly dependent.

Why? Put the vectors as columns of a matrix (so this matrix has n rows and more than n columns): there is only room for n pivots, but there is more than n columns. Hence, there is at least one column without pivot, and hence at least one free variable.

Note. While no work is required to see that these four vectors are linearly dependent, some work is needed to actually exhibit a dependence relation. Do that!

(d) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ **Solution.** These vectors are linearly independent.

Actually requires some work: $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \overset{R_3+R_1 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \overset{R_3-3R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

Example 86. Are the vectors $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$ linearly independent?

If they are dependent, exhibit a linear dependence relation.

Solution. The vectors are linearly independent if and only if the system

$$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$. By the usual steps of elimination, we find

$$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \overset{R_2+2R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix} \overset{R_3-2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there is a free variable, the solution $\boldsymbol{x} = \boldsymbol{0}$ is not unique. Hence, the vectors are linearly dependent.

Note that we didn't include the zero vector as the right-hand side to our matrix. Why was that OK?!

From the RREF, we can immediately read off the general solution: $x_3 = s$, $x_2 = -4s$, $x_1 = -5s$.

Hence, we have found the dependence relation $-5\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \boldsymbol{0}$.

(More generally, $-5s\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - 4s\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + s\begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \boldsymbol{0}$, but this is just scaled and contains no extra information.)

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Note that we already compiled some of the following statements in Theorem 63.

Theorem 87. Let A be an $n \times n$ matrix. The following statements are all equivalent:

[That is, if one of these is true, then so are all the others.]

A is invertible.

$\iff$ $\det(A) \neq 0$

$\iff$ The RREF of A is I_n.

$\iff$ (Any echelon form of) A has n pivots. (Easy to check!)

$\iff$ For every $\boldsymbol{b} \in \mathbb{R}^n$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

Why? If A is invertible then that unique solution is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. On the other hand, if we have a unique solution (even for just a single case $\boldsymbol{b}$), then we cannot have a free variable. This means that each of the n columns of A contains a pivot, and so we have n of them.

$\iff$ The system $A\boldsymbol{x} = \boldsymbol{0}$ has a unique solution.

Why? Because if the solution is unique, then there are no free variables: every column contains a pivot, and so A has n pivots.

[There is nothing special about $\boldsymbol{0}$ here; you could replace $\boldsymbol{0}$ with any specific vector in $\mathbb{R}^n$.]

$\iff$ The columns of A are linearly independent.

Why? This is the same as saying $A\boldsymbol{x} = \boldsymbol{0}$ only has the trivial solution.

$\iff$ For every $\boldsymbol{b} \in \mathbb{R}^n$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

Why? This is a weaker statement than the one above, which also guarantees each system to have only one solution. On the other hand, if $A\boldsymbol{x} = \boldsymbol{b}$ always has a solution, then that means in an echelon form of A there cannot be a zero row because otherwise a well-chosen $\boldsymbol{b}$ would create an inconsistency (see Theorem 19). But that means every row contains a pivot, and so A has n pivots.

$\iff$ The columns of A span all of $\mathbb{R}^n$.

Why? Because this is just different language for saying $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b} \in \mathbb{R}^n$. (To see that, recall that $A\boldsymbol{x}$ is a linear combination of the columns of A.)

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