Example 79. Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$. Compute:

(a) $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} =$

(b) $(AB)^T = \begin{bmatrix} & & \end{bmatrix}$

(c) $B^T A^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} =$

(d) $A^T B^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} =$ What's that fishy smell?

Theorem 80. Let A, B be matrices of appropriate size. Then:

- $(A^T)^T = A$ obvious!
- $(A + B)^T = A^T + B^T$ obvious!
- $(AB)^T = B^T A^T$ (illustrated by the previous example)
- $(A^T)^{-1} = (A^{-1})^T$ **Why?** Do you see how this follows from the previous item?
- $\det(A^T) = \det(A)$

Example 81. Let A and B be $n \times n$ matrices with $\det(A) = a$ and $\det(B) = b$. Simplify $\det(3A^T A B^2 A^{-1})$.

Solution. $\det(3A^T A B^2 A^{-1}) = 3^n \det(A^T A B^2 A^{-1}) = 3^n \det(A^T)\det(A)\det(B^2)\det(A^{-1}) = 3^n a b^2$

10 Linear independence

Definition 82. Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are **(linearly) dependent** if

$$x_1 \boldsymbol{v}_1 + x_2 \boldsymbol{v}_2 + \ldots + x_n \boldsymbol{v}_n = \mathbf{0}$$

for some x_i, not all zero. [This is then called a linear dependence relation.]

[There is always the trivial linear combination in which **all** coefficients are 0: $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$.]

Otherwise, the vectors are **(linearly) independent**.

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Example 83. Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ linearly independent?

Solution. We need to find out if

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has any solutions besides the trivial solution $x_1 = x_2 = x_3 = 0$. But that's just asking whether a linear system (which is obviously consistent; why?!) has a unique solution or whether there are infinitely many solutions.

We therefore eliminate:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right] \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 - R_1 \Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the echelon form, we see that the system is consistent (it had to be!) and that it has infinitely many solutions (because there is a free variable).

Hence, our three vectors are not linearly independent.

Exhibit a linear dependence relation among the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$.

Solution. We have already done the bulk of the work in the previous problem.

For a change, let us solve the system by back-substitution. $x_3 = s_1$ is free. Then, $x_2 = -2s_1$ and $x_1 = -x_2 + x_3 = 3s_1$. This means that

$$3s_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2s_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s_1 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a non-trivial linear combination of our three vectors which produces the zero vector.

Note that setting $s_1 = 1$ produces a nice linear combination, and that every other linear combination is just a multiple.

Example 84. With the minimum amount of work, decide whether the following vectors are linearly independent.

(a) $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \end{bmatrix}$ **Solution.** These vectors are linearly independent.

Put them as columns of a matrix, and notice that this matrix is already in echelon form...

(b) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}$ **Solution.** These vectors are linearly independent.

If they were dependent, then $x_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} = \mathbf{0}$. Since $x_1 \neq 0$ (why?), $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = -\frac{x_2}{x_1} \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}$ so that the second vector would be a multiple of the first. But it isn't! (Judging by the first entry, the second vector would have to be 3 times the first; but that clashes with the third entry.)

Moral. two vectors are linearly dependent $\iff$ one is a multiple of the other

(c) $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **Solution.** These vectors are linearly dependent.

For instance, $0 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a non-trivial dependence relation (the coefficients are 0 and 7).

Moral. Whenever the zero vector is involved, the vectors are linearly dependent.

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