Example 49. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$ whereas $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

If you know about the dot product, do you see a connection with the first case?

Example 50. $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

Note that this means that the system of equations $\begin{array}{rcrcr} x_1 & + & 4x_2 & = & 1 \\ 2x_1 & + & 5x_2 & = & -3 \\ 3x_1 & + & 6x_2 & = & 0 \end{array}$ can also be written as $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$.

[This was the motivation for introducing matrix-vector multiplication.]

In the same way, any system can be written as $A\boldsymbol{x} = \boldsymbol{b}$, where A is a matrix and $\boldsymbol{b}$ a vector.

Why do we care? First of all, it is concise! But also:

- The compactness sparks associations and ideas!
  - For instance, can we solve by *dividing* by A? $\boldsymbol{x} = A^{-1}\boldsymbol{b}$?
  - If $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{y} = 0$, then $A(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{b}$.
- Leads to matrix calculus and deeper understanding.
  - multiplying, inverting, or factoring matrices

Example 51. $\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

In other words, a product can be zero even though none of the factors is! That's another example illustrating that we need to be careful when extrapolating our intuition to matrix multiplication.

Example 52. Suppose A is $m \times n$ and B is $p \times q$. When does AB make sense? In that case, what are the dimensions of AB?

AB makes sense if $n = p$. In that case, AB is a $m \times q$ matrix.

Example 53.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ 4 & 17 & 6 \\ 7 & 29 & 9 \end{bmatrix}$$

Multiplication (on the right) with that “almost identity matrix” was performing the operation $C_2 + 3C_1 \Rightarrow C_2$.

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 13 & 17 & 21 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Multiplication (on the left) with the same matrix was performing the operation $R_1 + 3R_2 \Rightarrow R_1$.

First comment. This hints at a second interpretation of matrix multiplication: instead of taking linear combinations of columns of the first matrix, we can also take linear combinations of rows of the second matrix. (If you are already familiar with matrix multiplication, try to see what this exactly means while you do the WeBWorK exercises.)

Second comment. The row operations we are doing during Gaussian elimination can be realized by multiplying (on the left) with “almost identity matrices”.

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7 Matrix inverses

Throughout the discussion on matrix inverses, A is always a $n\times n$ matrix (a **square matrix**).

Example 54.

- $\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1\\ -2 & 3 \end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & -1\\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$
  Recall that on the RHS we have the **identity matrix**, usually denoted I or I_2 (since it's the 2×2 identity matrix here).

Definition 55. A matrix A is invertible if it has an **inverse**, denoted A^{-1}, satisfying

$$A^{-1}A=I \quad \text{and} \quad AA^{-1}=I.$$

- If one of the conditions is true, the other is automatically true.
- If it exists, the inverse is unique. And so A^{-1} refers to a specific matrix.

Example 56. Solve $\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}\boldsymbol{x}=\begin{bmatrix} 1\\ 2 \end{bmatrix}$ using the inverse of $\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}$.

Solution. From the previous example we known that $\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & -1\\ -2 & 3 \end{bmatrix}$.

We multiply $\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}\boldsymbol{x}=\begin{bmatrix} 1\\ 2 \end{bmatrix}$ on both sides with $\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}^{-1}$ to get

$$\underbrace{\begin{bmatrix} 1 & -1\\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}}_{=\left[\begin{smallmatrix} 1 & 0\\ 0 & 1 \end{smallmatrix}\right]}\boldsymbol{x}=\begin{bmatrix} 1 & -1\\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1\\ 2 \end{bmatrix},$$

which tells us that $\boldsymbol{x}=\begin{bmatrix} 1 & -1\\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1\\ 2 \end{bmatrix}=\begin{bmatrix} -1\\ 4 \end{bmatrix}$.

Theorem 57. If A is invertible, then the system $A\boldsymbol{x}=\boldsymbol{b}$ has the unique solution $\boldsymbol{x}=A^{-1}\boldsymbol{b}$.

Proof. As in the previous example, we just have to multiply both sides of $A\boldsymbol{x}=\boldsymbol{b}$ with A^{-1}. □

Example 58. How to find the inverse of $\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}$?

Solution. Let's call the inverse X. Then we are looking for X such that $AX=I$.
When solving $A\boldsymbol{x}=\boldsymbol{b}$, we do Gaussian elimination on $[\,A\,|\,\boldsymbol{b}\,]$. Once in RREF, we can read off $\boldsymbol{x}$.
Let's do the same and do Gaussian elimination on $[\,A\,|\,I\,]$:

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0\\ 2 & 1 & 0 & 1 \end{array}\right] \overset{R_2-\frac{2}{3}R_1\Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0\\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right] \overset{\substack{\frac{1}{3}R_1\Rightarrow R_1\\ 3R_2\Rightarrow R_2}}{\rightsquigarrow} \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0\\ 0 & 1 & -2 & 3 \end{array}\right] \overset{R_1-\frac{1}{3}R_2\Rightarrow R_1}{\rightsquigarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1\\ 0 & 1 & -2 & 3 \end{array}\right]$$

Indeed, we can now read off that $\begin{bmatrix} 3 & 1\\ 2 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & -1\\ -2 & 3 \end{bmatrix}$.

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