Sketch of Lecture 5

Tue, 8/30/2016

Definition 28. The **span** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ is the set of all linear combinations

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_m\boldsymbol{v}_m,$$

where $x_1, x_2, \ldots, x_m$ can be any real numbers. We write $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ for this set.

A **span** is what we will call a **vector space**. It's a set of vectors which can be added and scaled without leaving that set.

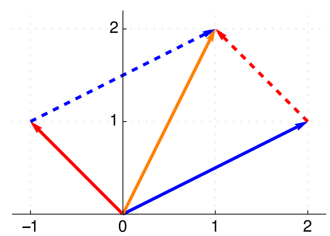
Example 29. Vectors in $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ include:

- $3\boldsymbol{v}_1 - \boldsymbol{v}_2 + 7\boldsymbol{v}_3$,
- $\boldsymbol{v}_2 + \boldsymbol{v}_3$,
- $\frac{1}{3}\boldsymbol{v}_2$,
- $\boldsymbol{0}$ (the zero vector).

Example 30. A vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$ can be represented by an arrow from the origin to the point (x_1, x_2).

But the position of the arrow doesn't matter. It can "start" anywhere.

Given $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, graph $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{x} + \boldsymbol{y}$, $2\boldsymbol{x}$.



Example 31. Is $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ in $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$?

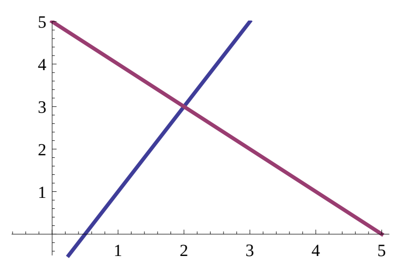
Solution. The question is, can we find x_1 and x_2 such that $x_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

This is the **vector form** of: $\begin{array}{rcl} 2x_1 - x_2 & = & 1 \\ x_1 + x_2 & = & 5 \end{array}$. We find $x_1 = 2$ and $x_2 = 3$. Indeed, $2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Example 32. We can think of the linear system $\begin{array}{rcl} 2x - y & = & 1 \\ x + y & = & 5 \end{array}$ in two different geometric ways.

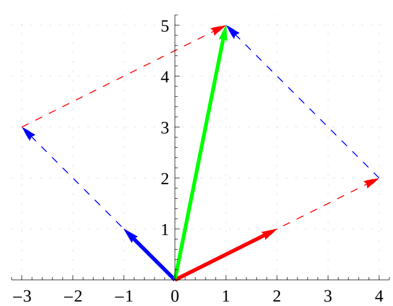
Row picture.

- Each equation defines a line in $\mathbb{R}^2$.
- Which points lie on the intersection of these lines?
- $(2, 3)$ is the (only) intersection of the two lines $2x - y = 1$ and $x + y = 5$.



Column picture.

- The system can be written as
$$x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$
- Which linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ produces $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$?
- $(2, 3)$ are the coeffs of the (only) such linear combination.



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Example 33. Is every vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$?

Solution. This is the same problem as earlier with $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ replaced with $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

The vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is in the span if and only if the system $\left[\begin{array}{rr|r} 2 & -1 & b_1 \\ 1 & 1 & b_2 \end{array}\right]$ is consistent.

One step of elimination: $\overset{R_2 - \frac{1}{2}R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{rr|r} 2 & -1 & b_1 \\ 0 & 3/2 & \ldots \end{array}\right]$. The ... is $b_1 - \frac{1}{2}b_2$ but, regardless, we see that the system is always consistent! (Why?!) Hence, the span contains all vectors. In other words, $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$.

Example 34. Is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$? If so, write $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Solution. By "staring", we see that $2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. So, yes!

Solution. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ if and only if there are x_1 and x_2 so that $x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

This is just the vector notation of the linear system with augmented matrix $\left[\begin{array}{rr|r} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$.

This system is already in echelon form. It is consistent (by Theorem 19). Hence, our vector is in the given span.

Moreover, by back substitution, we find $x_2 = 1$ and $x_1 = 2$. So, $2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Example 35. Is $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ in $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$? If so, write $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

Solution. As in the previous example, $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ is in $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$ if and only if $\left[\begin{array}{rr|r} 1 & 2 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 3 \end{array}\right]$ is consistent.

$$\left[\begin{array}{rr|r} 1 & 2 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 3 \end{array}\right] \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 - 2R_1 \Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{rr|r} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{array}\right] \overset{R_3 + 3R_2 \Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{rr|r} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

This system is consistent. Hence, our vector is in the given span.

By back substitution, we find $x_2 = -1$ and $x_1 = 2$. This means $2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$.

Example 36. Is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$? If so, write $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

Solution. Only the right-hand side of the linear system changed. The elimination steps are exactly the same!

[This is important for applications. It is often the case that a system needs to be solved for many different right-hand sides.]

$$\left[\begin{array}{rr|r} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 1 & 0 \end{array}\right] \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 - 2R_1 \Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -3 & -2 \end{array}\right] \overset{R_3 + 3R_2 \Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -5 \end{array}\right]$$

This system is inconsistent. Hence, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$.

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