3 Augmented matrix notation and echelon forms

$$\begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & -2 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \left[\begin{array}{rr|r} 2 & -4 & -2 \\ -1 & 3 & 3 \end{array}\right]$$

(augmented matrix)

Example 10. Let us solve the system in matrix notation.

$$\left[\begin{array}{rr|r} 2 & -4 & -2 \\ -1 & 3 & 3 \end{array}\right] \qquad \begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & -2 \\ -x_1 & + & 3x_2 & = & 3 \end{array}$$

$$\underset{\rightsquigarrow}{R_2+\tfrac{1}{2}R_1\Rightarrow R_2} \left[\begin{array}{rr|r} 2 & -4 & -2 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & -2 \\ & & x_2 & = & 2 \end{array}$$

Hence, $x_2 = 2$ and, by back-substitution, we find (from $2x_1 - 4\cdot 2 = -2$) that $x_1 = 3$. Alternatively, instead of back-substitution, we can also continue with row operations:

$$\underset{\rightsquigarrow}{R_1+4R_2\Rightarrow R_1} \left[\begin{array}{rr|r} 2 & 0 & 6 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{rcl} 2x_1 & & = 6 \\ & x_2 & = 2 \end{array}$$

$$\underset{\rightsquigarrow}{\tfrac{1}{2}R_1\Rightarrow R_1} \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{rcl} x_1 & & = 3 \\ & x_2 & = 2 \end{array}$$

In general, we are aiming for a stair-case shape in our approach. More precisely:

Definition 11. A matrix is in **echelon form** (or **row echelon form**) if:

- The leading entry of each row (i.e. the leftmost nonzero entry), referred to as a **pivot**, is in a column to the right of the leading entry of the row above it.

  And all zero rows are at the bottom of the matrix.

A matrix is in **(row)-reduced echelon form** (**RREF**) if, in addition to being in echelon form:

- Each pivot is 1, and it is the only nonzero entry in its column.

Example 12. A typical matrix in echelon form: ($*$ stands for any value, and $\blacksquare$ for any nonzero value.)

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the pivots are exactly the entries $\blacksquare$.
This matrix simplified to reduced echelon form:

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(The values of the entries $*$ are changing; we're only focusing on the structure.)

Armin Straub
straub@southalabama.edu

Example 13. Let us solve the following system:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & - & 4x_2 & - & 5x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

We write down the corresponding augmented matrix, and then simplify (the steps we are doing are referred to as row operations, and the whole process is called **Gaussian elimination**).

On the right-hand side, we record the corresponding linear system (just this one time and just for illustration).

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 3 & -4 & -5 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \qquad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & - & 4x_2 & - & 5x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\underset{\rightsquigarrow}{\substack{R_2-3R_1\Rightarrow R_2 \\ R_3+4R_1\Rightarrow R_3}} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \qquad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array}$$

$$\underset{\rightsquigarrow}{R_3+\tfrac{3}{2}R_2\Rightarrow R_3} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & & & x_3 & = & 3 \end{array}$$

The matrix is now in echelon form.

At this stage, we can solve the linear system by back-substitution:

$$x_3 = 3 \quad \rightsquigarrow \quad \begin{array}{c} 2x_2 - 8\cdot 3 = 8 \\ \Longleftrightarrow \; x_2 = 16 \end{array} \quad \rightsquigarrow \quad \begin{array}{c} x_1 - 2\cdot 16 + 1\cdot 3 = 0 \\ \Longleftrightarrow \; x_1 = 29 \end{array}$$

In conclusion, the system has the unique solution $(x_1, x_2, x_3) = (3, 16, 29)$.

Alternative. Instead of back-substitution, we can continue to simplify the system:

$$\underset{\rightsquigarrow}{\substack{R_1-R_3\Rightarrow R_1 \\ R_2+8R_3\Rightarrow R_2}} \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & 2x_2 & & & = & 32 \\ & & & & x_3 & = & 3 \end{array}$$

$$\underset{\rightsquigarrow}{\substack{R_1+R_2\Rightarrow R_1 \\ \frac{1}{2}R_2\Rightarrow R_2}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{rcl} x_1 & = & 29 \\ x_2 & = & 16 \\ x_3 & = & 3 \end{array}$$

The matrix is now in RREF.

The corresponding linear system is so simple, that we can just read off the solution $(x_1, x_2, x_3) = (29, 16, 3)$. (By the way, do you see how the original system is related to the one from last class?)

Example 14. (Exercise!) Consider, again, the linear system:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 1 \\ -x_1 & - & x_2 & + & 2x_3 & = & 1 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 5 \end{array}$$

Write down the corresponding augmented matrix.

Then, determine its RREF using Gaussian elimination. Read off the solution to the linear system (and compare with your previous result).

Armin Straub
straub@southalabama.edu