Preparing for Midterm #2

MATH 237 — Linear Algebra I
Thursday, Nov 12

Please print your name:

Computational part

Problem 1. Evaluate the following determinants. [Real computations only necessary for the last two.]

(a) $\begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{vmatrix}$

(e) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$

(f) $\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix}$

Solution.

(a) $\begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{vmatrix} = 0$ because the columns are not linearly independent. (Column one and two are the same.)

(b) $\begin{vmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot 2 \cdot 6 = 12$

(c) $\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} = (-1)(-1)(-1) = -1$ because it takes three row interchanges ($R_1 \Leftrightarrow R_6$, $R_2 \Leftrightarrow R_5$, $R_3 \Leftrightarrow R_4$) to transform this matrix to the 6×6 identity matrix.

(d) $\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{vmatrix} = 0$ because the matrix is clearly not invertible. (Look at the last column!)

(e) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 - 3R_1 \Rightarrow R_3}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & -4 & -8 \end{vmatrix} \overset{R_3 - 4R_2 \Rightarrow R_3}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -8 \end{vmatrix} = 1 \cdot (-1) \cdot (-8) = 8$

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(f) $\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix} \overset{\substack{R_2-2R_1\Rightarrow R_2 \\ R_3+R_1\Rightarrow R_3}}{=} \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix} \overset{R_4+2R_2\Rightarrow R_4}{=} \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{vmatrix} \overset{R_4+\frac{5}{2}R_3\Rightarrow R_4}{=} \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{vmatrix}$

$= 1 \cdot (-1) \cdot (-2) \cdot 10 = 20$ □

Problem 2. Find a basis for $\mathrm{col}(A), \mathrm{row}(A), \mathrm{null}(A)$ with

(a) $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -1 & -2 & -1 & -1 & -3 \\ 2 & 4 & 0 & -6 & 7 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c) $A = [\, 1 \;\; 2 \;\; 3 \,]$

Solution.

(a) Our first step is to bring A into RREF (just an echelon form would be enough, but then we would need to back-substitute when solving $A\boldsymbol{x} = \mathbf{0}$ for $\mathrm{null}(A)$):

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -1 & -2 & -1 & -1 & -3 \\ 2 & 4 & 0 & -6 & 7 \end{bmatrix} \underset{\text{do it!}}{\overset{\text{RREF}}{\rightsquigarrow}} \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- A basis for $\mathrm{col}(A)$ is: $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$. $(\dim \mathrm{col}(A) = 3)$

- A basis for $\mathrm{row}(A)$ is: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. $(\dim \mathrm{row}(A) = 3)$

- $x_2 = s_1$ and $x_4 = s_2$ are our free variables. The general solution to $A\boldsymbol{x} = \mathbf{0}$ is:

$$\boldsymbol{x} = \begin{bmatrix} -2s_1 + 3s_2 \\ s_1 \\ -4s_2 \\ s_2 \\ 0 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

  Hence, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ is a basis for $\mathrm{null}(A)$. $(\dim \mathrm{null}(A) = 2)$

(b) A basis for $\mathrm{col}(A)$ is: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

A basis for $\mathrm{row}(A)$ is: $[\, 1 \,]$.

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null$(A) = \{[\ 0\ ]\}$ (only the trivial solution), which has dimension 0 and therefore a basis with 0 vectors (that is, a/the basis is the empty set $\emptyset$).

(c) A basis for col(A) is: $[\ 1\ ]$.

A basis for row(A) is: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

The general solution to $A\boldsymbol{x} = \boldsymbol{0}$ is (note that A is in RREF already) $\boldsymbol{x} = \begin{bmatrix} -2s_1 - 3s_2 \\ s_1 \\ s_2 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

Hence, a basis for null(A) is: $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. □

Problem 3.

(a) Is $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} : a - b = c,\ a - d = e \right\}$ a vector space? If yes, find a basis.

(b) Is $W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ a vector space? If yes, find a basis.

(c) Is $W = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ a vector space? If yes, find a basis.

(d) Is $W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ a vector space? If yes, find a basis.

Solution.

(a) The matrix form of the linear equations $a - b = c,\ a - d = e$ is $\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This means that $W = \text{null}\left(\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \end{bmatrix} \right)$.

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \end{bmatrix} \overset{R_2 - R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 \end{bmatrix} \overset{R_1 + R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

The general solution of our system is $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} s_2 + s_3 \\ -s_1 + s_2 + s_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = s_1 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

A basis for W is $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) Yes, W is a vector space. It has dimension 0 and therefore a basis with 0 vectors (that is, a/the basis is the empty set $\emptyset$).

(c) No, W is not a vector space, because $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$.

(d) No, W is not a vector space. For instance, it contains $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ but not $-1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. □

Problem 4. Consider $H = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$.

(a) Give a basis for H. What is the dimension of H?

(b) Determine whether the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is in H. What about the vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$?

(c) Extend the basis of H to a basis of $\mathbb{R}^3$.

Solution.

(a) Clearly, $H = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$.

Moreover, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are a basis for H (because these two vectors are linearly independent).

In particular, $\dim H = 2$.

(b) We need to solve $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

Let us do both at the same time (by working with two right-hand sides at once):

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \end{array}\right] \underset{\rightsquigarrow}{R_2 - R_1 \Rightarrow R_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 \end{array}\right] \underset{\rightsquigarrow}{R_3 - R_2 \Rightarrow R_3} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The first equation is inconsistent and so $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in H.

The first equation is consistent and so $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ is in H.

(c) We need to add a third vector to our basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ of H. In the previous part, we found that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in H.

In other words, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are linearly independent.

It follows that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are a basis for $\mathbb{R}^3$. □

Problem 5. Is it true that $\operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix}\right\}$?

Solution. Let $V = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}\right\}$ and $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix}\right\}$.

We check that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \in V$ and $\begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \in V$.

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(This follows, as in the previous problem, from $\left[\begin{array}{rr|rr} 1 & 0 & 1 & 2 \\ -1 & -2 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right] \overset{\substack{R_2+R_1\Rightarrow R_2 \\ R_3-R_1\Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{rr|rr} 1 & 0 & 1 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right] \overset{R_4+\frac{1}{2}R_2\Rightarrow R_4}{\rightsquigarrow} \left[\begin{array}{rr|rr} 1 & 0 & 1 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, because for both right-hand sides the system is consistent.)

Since these two vectors span W, this implies that W is a subspace of V.

But both spaces have dimension 2, and so they must be equal: $V = W$. □

Short answer part

Problem 6. Let A be a 5×4 matrix. Suppose that the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has the solution set

$$\left\{ \begin{bmatrix} 1-c+d \\ c \\ 3-2d \\ d \end{bmatrix} : c, d \text{ in } \mathbb{R} \right\}.$$

(a) Give a basis for the null space of A.

(b) What is the rank of A?

Solution.

(a) Observe that $\begin{bmatrix} 1-c+d \\ c \\ 3-2d \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

Here, $\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$ is a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ and $c\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ is the general solution to $A\boldsymbol{x} = \boldsymbol{0}$.

In particular, $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ are a basis for $\operatorname{null}(A)$.

(b) $\operatorname{rank}(A) = 4 - 2 = 2$ because A has 4 columns and we know that 2 of them correspond to free variables. □

Problem 7. In each case, write down a precise definition or answer.

(a) What is a vector space?

(b) What is the rank of a matrix?

(c) What does it mean for vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ from a vector space to be linearly independent?

(d) List the elementary row operations.

(e) What does it mean for vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ to be a basis for a vector space V?

Solution.

(a) A vector space is a set V of vectors that can be written as a span (that is, $V = \text{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots\}$ for a bunch of vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$).

An alternative, more abstract, definition is: A vector space is a set V of vectors such that

- if $\boldsymbol{v}, \boldsymbol{w} \in V$, then $\boldsymbol{v} + \boldsymbol{w} \in V$, [closed under addition]
- if $\boldsymbol{v} \in V$ and $r \in \mathbb{R}$, then $r\boldsymbol{v} \in V$. [closed under scalar multiplication]

(b) The rank of a matrix is the number of pivots in an echelon form.

Alternatively: The rank of a matrix is the dimension of its column space. (Or, row space.)

(c) Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent if the only solution to

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_n\boldsymbol{v}_n = \mathbf{0}$$

is the trivial one ($x_1 = x_2 = \ldots = x_n = 0$).

(d) The elementary row operations are:

- *(replacement)* $R_j - \lambda R_i \Rightarrow R_j$
- *(swap two rows)* $R_j \Leftrightarrow R_i$
- *(scaling)* $\lambda R_i \Rightarrow R_i$

(e) The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are a basis for V, if $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are linearly independent and $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$.

□

Problem 8. Let A be a $n \times n$ matrix. List at least five other statements which are equivalent to the statement "A is invertible".

Solution. Here are a few possibilities:

A is invertible.
$\Longleftrightarrow$ The RREF of A is I_n.
$\Longleftrightarrow$ A has n pivots.
$\Longleftrightarrow$ $\text{rank}(A) = 0$
$\Longleftrightarrow$ For every $\boldsymbol{b} \in \mathbb{R}^n$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.
$\Longleftrightarrow$ The system $A\boldsymbol{x} = \mathbf{0}$ has a unique solution.
$\Longleftrightarrow$ $\dim \text{null}(A) = 0$
$\Longleftrightarrow$ The columns of A are linearly independent.
$\Longleftrightarrow$ The rows of A are linearly independent.
$\Longleftrightarrow$ For every $\boldsymbol{b} \in \mathbb{R}^n$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.
$\Longleftrightarrow$ The columns of A span all of $\mathbb{R}^n$.
$\Longleftrightarrow$ $\dim \text{col}(A) = n$
$\Longleftrightarrow$ The rows of A span all of $\mathbb{R}^n$.
$\Longleftrightarrow$ $\dim \text{row}(A) = n$
$\Longleftrightarrow$ $\det(A) \neq 0$

□

Problem 9.

(a) Suppose V and W are subspaces of $\mathbb{R}^n$, and that $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a basis for V, and $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ is a basis for W. What can you say about $\dim U$ with $U = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$?

(b) Let A be a 4×3 matrix, whose row space has dimension 2. What is the dimension of null(A)?

(c) Let A be a 3×3 matrix, whose column space has dimension 3. If $\boldsymbol{b}$ is a vector in $\mathbb{R}^3$, what can you say about the number of solutions to the equation $A\boldsymbol{x} = \boldsymbol{b}$?

(d) Let A be a 3×3 matrix, whose column space has dimension 2. What can you say about $\det(A)$?

Solution.

(a) $\dim U \in \{3, 4, 5\}$

(b) $\dim \operatorname{null}(A) = 3 - 2 = 1$

(c) If A is a 3×3 matrix, whose column space has dimension 3, then A is invertible.

Therefore, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for any $\boldsymbol{b}$.

(d) If A is a 3×3 matrix, whose column space has dimension 2, then A is not invertible.

Therefore, $\det(A) = 0$. □

Problem 10. True or false?

(a) Every vector space has a basis.

(b) The zero vector can never be a basis vector.

(c) Every set of linearly independent vectors in V can be extended to a basis of V.

(d) col(A) and row(A) always have the same dimension.

(e) If B is the RREF of A, then we always have $\operatorname{col}(A) = \operatorname{col}(B)$.

(f) If B is the RREF of A, then we always have $\operatorname{row}(A) = \operatorname{row}(B)$.

(g) If a subspace V of $\mathbb{R}^3$ contains three linearly independent vectors, then always $V = \mathbb{R}^3$.

(h) There are matrices A such that null(A) is the empty set.

Solution.

(a) True. In fact, for all the spaces we can get our hands on, we know how to compute a basis.

[In the case of very infinite-dimensional spaces, this becomes "the axiom of choice".]

(b) True. A set of vector that includes the zero vector can never be linearly independent.

(c) True. We just keep adding missing vectors from V to the initial set of linearly independent vectors until we span all of V. (If V has dimension d, then this process of adding vectors has to stop once we have a total of d vectors.)

(d) True.

(e) False. Elementary row operations do not preserve column spaces (except by accident).

For instance, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \overset{R_1 \Leftrightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but $\operatorname{col}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \neq \operatorname{col}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$.

(f) True. Elementary row operations do preserve the row space.

(g) True. Three linearly independent vectors in $\mathbb{R}^3$ automatically form a basis of $\mathbb{R}^3$.

(h) False. null(A) always contains at least the zero vector (the trivial solution to $A\boldsymbol{x} = \boldsymbol{0}$). □