Review 154.

- **Eigenvector** equation: $A\boldsymbol{x} = \lambda \boldsymbol{x} \iff (A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$

  λ is an **eigenvalue** of $A \iff \underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0.$

- An $n \times n$ matrix A has up to n different eigenvalues λ.
  - The **eigenspace** of λ is $\operatorname{null}(A - \lambda I)$.

    That is, all eigenvectors of A with eigenvalue λ.

    Since $A - \lambda I$ has determinant 0, $\operatorname{null}(A - \lambda I)$ always has dimension at least 1.
  - If λ has **multiplicity** m (see examples below), then A has up to m eigenvectors for λ.

    At least one eigenvector is guaranteed (because $\det(A - \lambda I) = 0$).

Example 155. Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ as well as bases for the corresponding eigenspaces.

Solution. The characteristic polynomial is $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1$.

The roots of this polynomial, that is, the eigenvalues are $\lambda = 1$ and $\lambda = 3$.

- For $\lambda = 1$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
- For $\lambda = 3$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example 156. Find the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ as well as bases for the corresponding eigenspaces.

Solution. By expanding by the first row, we find that the characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(3-\lambda)^2 - 1] = (2-\lambda)(\lambda-2)(\lambda-4).$$

Since $\lambda = 2$ is a double root, it has **(algebraic) multiplicity** 2.

Hence, the eigenvalues are $\lambda = 2$ (with multiplicity 2) and $\lambda = 4$.

- For $\lambda = 4$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.
- For $\lambda = 2$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Example 157. Determine the eigenvalues as well as corresponding eigenspaces.

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: char poly $(1-\lambda)^2$; eigenvalue $\lambda = 1$ (with multiplicity 2) and eigenspace $\mathbb{R}^2$
- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: char poly $(1-\lambda)^2$; eigenvalue $\lambda = 1$ (with multiplicity 2) and eigenspace $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

  This illustrates that an eigenspace can have dimension less than the multiplicity of the eigenvalue.
- $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$: char poly λ^2; eigenvalue $\lambda = 0$ (with multiplicity 2) and eigenspace $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$