**Example 137.** What's **wrong** in the following "calculation"?!

$$\det(A^{-1}) = \det\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{ad-bc}(da-(-b)(-c)) = 1$$

**Solution.** The corrected calculation is: $\det\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{(ad-bc)^2}(da-(-b)(-c)) = \frac{1}{ad-bc}$

**Note.** It is always true that $\det(A^{-1}) = \frac{1}{\det(A)}$.

**Remark.** If you are still confused about the above mistake: note that $\det\left(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 4$ (not $2$).

**Example 138.** Suppose $A$ is a $3\times 3$ matrix with $\det(A) = -2$. What is $\det(10A)$?

**Solution.** $\det(10A) = 10^3 \cdot (-2) = -2000$ (because $A$ has $3$ rows, each of which gets multiplied with $10$).

The following important properties follow from the behaviour under row operations.

- $\det(A) = 0 \iff A$ is not invertible

  Why? Because $\det(A) = 0$ if only if, in an echelon form, a diagonal entry is zero (that is, a pivot is missing).
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(A^T) = \det(A)$

**Example 139.** Let $A$ be an $n\times n$ matrix with $\det(A) = d$. Simplify $\det(A^3)$ and $\det(3A)$.

**Solution.** $\det(A^3) = \det(A\cdot A\cdot A) = \det(A)\det(A)\det(A) = d^3$ and $\det(3A) = 3^n d$.

## A "bad" way to compute determinants

**Example 140.** Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

**Solution.** We expand by the first row:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\cdot\begin{vmatrix} + & & \\ & -1 & 2 \\ & 0 & 1 \end{vmatrix} - 2\cdot\begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} & & + \\ 3 & -1 & \\ 2 & 0 & \end{vmatrix}$$

$$\overset{\text{i.e.}}{=} 1\cdot\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\cdot\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 1\cdot(-1) - 2\cdot(-1) + 0 = 1$$

Each term in the cofactor expansion is $\pm 1$ times an entry times a smaller determinant (row and column of entry deleted).

The $\pm 1$ is assigned to each entry according to $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$.

**Solution.** We expand by the second column:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2\cdot\begin{vmatrix} & - & \\ 3 & & 2 \\ 2 & & 1 \end{vmatrix} + (-1)\cdot\begin{vmatrix} 1 & & 0 \\ & + & \\ 2 & & 1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & & 0 \\ 3 & & 2 \\ & - & \end{vmatrix}$$

$$= -2\cdot(-1) + (-1)\cdot 1 - 0 = 1$$

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Why is the method of cofactor expansion not practical?

Because to compute a large $n \times n$ determinant,

- one reduces to $n$ determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

In the end, we have $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ many numbers to add.

WAY TOO MUCH WORK! Already $25! = 15511210043330985984000000 \approx 1.55 \cdot 10^{25}$.

Context: today's fastest computer, Tianhe-2, runs at $34$ petaflops ($3.4 \cdot 10^{16}$ op's per second).

By the way: "fastest" is measured by doing Gaussian elimination!

# Linear transformations

Throughout, $V$ and $W$ are vector spaces.

**Definition 141.** A map $T\colon V \to W$ is a **linear transformation** if

$$T(c\boldsymbol{x} + d\boldsymbol{y}) = cT(\boldsymbol{x}) + dT(\boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \text{ in } V \text{ and all } c, d \text{ in } \mathbb{R}.$$

In other words, a linear transformation respects addition and scaling:

- $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$
- $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$

It also sends the zero vector in $V$ to the zero vector in $W$:

- $T(\boldsymbol{0}) = \boldsymbol{0}$ [because $T(\boldsymbol{0}) = T(0 \cdot \boldsymbol{0}) = 0 \cdot T(\boldsymbol{0}) = \boldsymbol{0}$]

**Example 142.** Let $A$ be an $m \times n$ matrix.

Then the map $T(\boldsymbol{x}) = A\boldsymbol{x}$ is a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

Why?

Because matrix multiplication is linear:

$A(c\boldsymbol{x} + d\boldsymbol{y}) = cA\boldsymbol{x} + dA\boldsymbol{y}$

The LHS is $T(c\boldsymbol{x} + d\boldsymbol{y})$ and the RHS is $cT(\boldsymbol{x}) + dT(\boldsymbol{y})$.

## Important geometric examples

We consider some linear maps $\mathbb{R}^2 \to \mathbb{R}^2$, which are defined by matrix multiplication, that is, by $\boldsymbol{x} \mapsto A\boldsymbol{x}$.

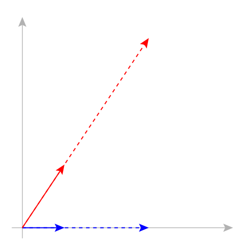
In fact: all linear maps $\mathbb{R}^n \to \mathbb{R}^m$ are given by $\boldsymbol{x} \mapsto A\boldsymbol{x}$, for some matrix $A$.

**Example 143.**

The matrix $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto c\begin{bmatrix} x \\ y \end{bmatrix}$, i.e.

... stretches every vector in $\mathbb{R}^2$ by the same factor $c$.



**Example 144.**

The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

... gives the map $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$, i.e.

... reflects every vector in $\mathbb{R}^2$ through the line $y = x$.