Example 127.

(a) Is $W = \left\{ \begin{bmatrix} a \\ a-b \\ 2b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ a vector space? If yes, find a basis.

Solution. Since $\begin{bmatrix} a \\ a-b \\ 2b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, we see that $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}$.

In particular, W is a vector space with basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ (because these two vectors are independent).

Note. Geometrically, W is a plane (through the origin) in $\mathbb{R}^3$.

(b) Is $W = \left\{ \begin{bmatrix} a \\ a-b \\ 2 \end{bmatrix} : a, b \in \mathbb{R} \right\}$ a vector space? If yes, find a basis.

Solution. W is not a vector space because $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$.

Note. As in the previous case, we can write $W = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$.

Hence, geometrically, W is still a plane in $\mathbb{R}^3$, but not through the origin.

(c) Is $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b = 2c \right\}$ a vector space? If yes, find a basis.

Solution. Writing $a + b = 2c$ as $a + b - 2c = 0$, we see that $W = \operatorname{null}([\begin{array}{ccc} 1 & 1 & -2 \end{array}])$.

In particular, W is a vector space. Since $A = [\begin{array}{ccc} 1 & 1 & -2 \end{array}]$ is already in RREF, we can read off the general solution $\boldsymbol{x} = \begin{bmatrix} -s_1 + 2s_2 \\ s_1 \\ s_2 \end{bmatrix}$ to $A\boldsymbol{x} = \boldsymbol{0}$. Hence, a basis for W is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(d) Is $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b = 2 \right\}$ a vector space? If yes, find a basis.

Solution. W is not a vector space because $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$.

Note. The equation $a + b + 0c = 2$ is inhomogeneous, with particular solution $\boldsymbol{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

We can therefore write $W = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \operatorname{null}([\begin{array}{ccc} 1 & 1 & 0 \end{array}])$.

Hence, geometrically, W is still a plane in $\mathbb{R}^3$, but not through the origin.

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Determinants

Example 128. Describe $\mathrm{col}(A), \mathrm{row}(A), \mathrm{null}(A)$ if A is an invertible $n \times n$ matrix.

Solution. Recall that A is invertible if and only if its RREF is I_n, the $n \times n$ identity matrix.
Therefore, $\dim \mathrm{col}(A) = n$, $\dim \mathrm{row}(A) = n$, $\dim \mathrm{null}(A) = 0$.
Consequently, $\mathrm{col}(A) = \mathbb{R}^n$, $\mathrm{row}(A) = \mathbb{R}^n$, $\mathrm{null}(A) = \{\mathbf{0}\}$.

For the next few lectures, all matrices are square!

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The **determinant** of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,
- a 1×1 matrix is $\det([\, a \,]) = a$.

Goal: A is invertible $\iff$ $\det(A) \neq 0$

We will write both $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for the determinant.

Definition 129. The **determinant** is characterized by:

- the normalization $\det I = 1$,
- and how it is affected by elementary row operations:
  - **(replacement)** $R_j - \lambda R_i \Rightarrow R_j$ does not change the determinant.
  - **(swap two rows)** $R_j \Leftrightarrow R_i$ reverses the sign of the determinant.
  - **(scaling)** $\lambda R_i \Rightarrow R_i$ multiplies the determinant by λ.

Example 130. Compute $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{vmatrix}$.

Solution. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{vmatrix} \overset{\frac{1}{2}R_2 \Rightarrow R_2}{=} 2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{vmatrix} \overset{\frac{1}{7}R_3 \Rightarrow R_3}{=} 14\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 14$

Example 131. Compute $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{vmatrix} \overset{\frac{1}{2}R_2 \Rightarrow R_2}{=} 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \end{vmatrix} \overset{\frac{1}{7}R_3 \Rightarrow R_3}{=} 14\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \overset{\substack{R_1 - 3R_3 \Rightarrow R_1 \\ R_2 - 2R_3 \Rightarrow R_2}}{=} 14\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \overset{R_1 - 2R_2 \Rightarrow R_1}{=} 14\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 14$$

The determinant of a triangular matrix is the product of the diagonal entries.

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