Example 125. Suppose V and W are subspaces of $\mathbb{R}^n$, and that $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a basis for V, and $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ is a basis for W. What can you say about $\dim U$ with $U = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$?

Solution. $\dim U \in \{3, 4, 5\}$ Why?

Recall that a basis of V is a list of vectors in V, which span V and which are linearly independent. The following is a rephrasing of that:

Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ be a basis of V. Every vector in V can be written uniquely as a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$.

Why? "can be written" because a basis spans V. "uniquely" because basis vectors are linearly independent.

Example 126.

(a) Is $U = \left\{ \begin{bmatrix} a - 2b \\ a + b \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ a vector space?

Solution. Since $\begin{bmatrix} a - 2b \\ a + b \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, we have $U = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

In particular, U is a vector space.

(b) Are $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ in U?

Solution. Clearly, $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ are a basis for U.

- Since $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{array}\right]$ is inconsistent (can you see it? If not, continue to an echelon form), the vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is not in U.
- On the other hand, $\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{array}\right]$ has the unique solution $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which shows that the vector $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is in U and that, in our basis, it can be uniquely written as $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 1\boldsymbol{v}_1 + 1\boldsymbol{v}_2$.

Important comment. This illustrates that every vector in U can be (uniquely) represented by its coefficients when expressing it in our basis of U. Since $\dim U = 2$, these are two coefficients (here, 1 and 1) which determine every vector in U. This allows us to work with U as if it was $\mathbb{R}^2$.

This is especially important, because vector spaces can be very abstract. For instance, V could consists of certain polynomials. But, once we pick a basis for V, we can start working with V as if it was one of the familiar spaces $\mathbb{R}^n$ (assuming that V has finite dimension).

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(c) Is $W = \left\{ \begin{bmatrix} a-2b \\ a+b \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$ a vector space?

Solution. Note that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$. (Why?) However, every span contains the zero vector. (Again, why?)

Hence, W is not a vector space.

Every vector space contains the/a zero vector.

(d) Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \geqslant 0 \right\}$ a vector space?

Solution. Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$. Therefore, if W is a vector space, then any vector in $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ has to be in W. However, for instance, $-1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not in W. Hence, W is not a vector space.

(e) Is $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + 2c = 1 \right\}$ a vector space?

Solution. Note that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$. (Why?) Hence, W is not a vector space.

(f) Is $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + 2c = 0 \right\}$ a vector space?

Solution. Note that W contains precisely the solutions $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to the linear equation $\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$.

In other words, W is nothing but $\operatorname{null}(\begin{bmatrix} 1 & 1 & 2 \end{bmatrix})$.

In particular, W is a vector space.

We have introduced vector spaces as sets of vectors that can be written as spans. Here is a slightly more abstract characterization of vector spaces:

> A vector space is a nonempty set V of vectors such that
> - if $\boldsymbol{v}, \boldsymbol{w} \in V$, then $\boldsymbol{v} + \boldsymbol{w} \in V$, [closed under addition]
> - if $\boldsymbol{v} \in V$ and $r \in \mathbb{R}$, then $r\boldsymbol{v} \in V$. [closed under scalar multiplication]
>
> In particular, every vector space has to contain the zero vector. (Why?)

Note that the two requirements can be combined by saying that, if $\boldsymbol{v}, \boldsymbol{w} \in V$, then any linear combination $r\boldsymbol{v} + s\boldsymbol{w} \in V$. And we are back at saying that V should be a span ($V = \operatorname{span} V$).