Review 117. Let A be a matrix and B an echelon form of A.

(a) The columns of A corresponding to the pivot columns of B form a basis for $\mathrm{col}(A)$.

(b) The nonzero rows of B form a basis for $\mathrm{row}(A)$.

(c) In particular, the dimensions of $\mathrm{col}(A)$ and $\mathrm{row}(A)$ are both equal to the number of pivots.

Example 118. Let $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find a basis for $\mathrm{col}(A)$ and $\mathrm{row}(A)$.

Solution. Clearly, a basis for $\mathrm{col}(A)$ is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. The dimension is 1.

Again, clearly, a basis for $\mathrm{row}(A) = \mathrm{span}\{[\ 1\ ], [\ 2\ ], [\ 3\ ]\}$ is $[\ 1\ ]$. The dimension is also 1.

[Recall that the last part of Theorem 116 tells us that the dimensions of $\mathrm{col}(A)$ and $\mathrm{row}(A)$ always agree.]

Example 119. Find a basis for $\mathrm{col}(A)$ with $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$.

Note that we have two choices because we can

- use Theorem 116 (a) directly, or
- use Theorem 116 (b) and $\mathrm{col}(A) = \mathrm{row}(A^T)$.

The first option will produce a basis from a subset of the original columns, while the second option will introduce new vectors (with some zeros). The amount of computation is the same.

Solution. Since

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 - R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

a basis for $\mathrm{col}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

[Recall that bases are not at all unique. For instance, now that we know that $\mathrm{col}(A)$ is 2-dimensional, we see that any pair of its columns would form a basis (because every pair of columns is linearly independent).]

Solution. Note that $\mathrm{col}(A) = \mathrm{row}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \end{bmatrix}\right)$. Since

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \end{bmatrix} \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 + R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

a basis for $\mathrm{col}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

[Note that, in this case, we get a basis that is not taken from the columns of A. Here, we can still see how it is related to the basis we obtained earlier: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.]

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Bases for null spaces

To find a basis for $\operatorname{null}(A)$:

- find the parametric form of the solutions to $A\boldsymbol{x} = \mathbf{0}$,
- express solutions $\boldsymbol{x}$ as a linear combination of vectors with the free variables as coefficients;
- these vectors form a basis of $\operatorname{null}(A)$.

In particular, the dimension of $\operatorname{null}(A)$ equals the number of free variables.

Example 120. Find a basis for $\operatorname{null}(A)$ with $A = \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 2 & 4 & 5 & 0 & 1 \end{bmatrix}$.

Solution. We eliminate!

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 2 & 4 & 5 & 0 & 1 \end{bmatrix} \overset{R_2 - 2R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix} \overset{R_1 - 2R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 5 & 13 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix}$$

$x_2 = s_1$, $x_4 = s_2$ and $x_5 = s_3$ are our free variables. The solutions to $A\boldsymbol{x} = \mathbf{0}$ are:

$$\boldsymbol{x} = \begin{bmatrix} -2s_1 - 5s_2 - 13s_3 \\ s_1 \\ 2s_2 + 5s_3 \\ s_2 \\ s_3 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$.

These vectors are clearly independent.

If you don't see it, do compute an echelon form! (permute first and third row to the bottom)

Better yet: note that the first vector corresponds to the solution with $s_1 = 1$ and the other free variables $s_2 = 0$, $s_3 = 0$. The second vector corresponds to the solution with $s_2 = 1$ and the other free variables $s_1 = 0$, $s_3 = 0$. The third vector ...

Hence, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $\operatorname{null}(A)$.