Bases for column and row spaces

Example 112. Find a basis for $\mathrm{col}(A)$ with $A = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$.

Solution. Obviously, $\mathrm{col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$.
The vectors $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ are also clearly independent, so they form a basis for $\mathrm{col}(A)$.

Solution. We can also apply the recipe from Theorem 109. Since A is already in echelon form, we see directly that columns 2 and 4 correspond to free variables. Columns 1 and 3, that is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, are therefore selected as a basis.

Example 113. Find a basis for $\mathrm{col}(A)$ with $A = \begin{bmatrix} 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 1 & -3 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$.

Solution. Since A is already in echelon form, we can directly apply the recipe from Theorem 109.

Columns 3 and 5 correspond to a free variable, so a basis for $\mathrm{col}(A)$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$.

[Note that many other choices would also lead to a basis: for instance, columns $1, 3, 4$ also form a basis. Can you see that? Likewise, columns $2, 3, 5$ form a basis as well.]

Example 114. Find a basis and the dimension of $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Solution. Is $\dim W = 4$? No, because the third vector is the sum of the first two.

Suppose we did not notice...

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 3 & 1 \end{bmatrix} \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_4 - 3R_1 \Rightarrow R_4}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & 1 \end{bmatrix} \overset{\substack{\text{permute} \\ \text{rows}}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{R_3 + 3R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{R_4 - \frac{1}{4}R_3 \Rightarrow R_4}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Not a pivot in every column, hence the 4 vectors are dependent.

Hence, a basis for W is $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\dim W = 3$.

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Example 115. Find bases for $\operatorname{col}(A)$ and $\operatorname{row}(A)$ with $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Solution. Obviously, $\operatorname{col}(A) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is 1-dimensional with basis $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
Similarly, $\operatorname{row}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is 1-dimensional with basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

During elimination, we usually do **row operations**. For instance, here, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \overset{R_1 \Leftrightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

- Note that the column spaces changed: $\operatorname{col}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \neq \operatorname{col}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$
- But the row spaces did not change: $\operatorname{row}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \operatorname{row}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$

Row operations preserve the row space but (generally) not the column space.

Theorem 116. Let A be a matrix and B an echelon form of A.
(a) The columns of A corresponding to the pivot columns of B form a basis for $\operatorname{col}(A)$.
(b) The nonzero rows of B form a basis for $\operatorname{row}(A)$.
(c) In particular, the dimensions of $\operatorname{col}(A)$ and $\operatorname{row}(A)$ are both equal to the number of pivots.

Just for fun

Q: *How fast can we solve N linear equations in N unknowns?*

Estimated cost of Gaussian elimination:

$$\begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}$$

- to create the zeros below the pivot:
  $\Longrightarrow$ on the order of N^2 operations
- if there is N pivots:
  $\Longrightarrow$ on the order of $N \cdot N^2 = N^3$ op's

- A more careful count places the cost at $\sim \frac{1}{3}N^3$ operations.
- For large N, it is only the N^3 that matters.

  It says that if $N \to 10N$ then we have to work 1000 times as hard.

That's not optimal! We can do better than Gaussian elimination:

- Strassen algorithm (1969): $N^{\log_2 7} = N^{2.807}$
- Coppersmith–Winograd algorithm (1990): $N^{2.375}$
- ... Stothers–Williams–Le Gall (2014): $N^{2.373}$

Is $N^{2+(\text{a tiny bit})}$ possible? **We have no idea!** (better is impossible; why?)

Good news for applications:

- Matrices typically have lots of structure and zeros

  which makes solving so much faster.

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