Review. spaces, basis, dimension, $\operatorname{col}(A)$, $\operatorname{row}(A)$, $\operatorname{null}(A)$

Example 104.

(a) Consider $V = \operatorname{col}\left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\right)$. By definition, $V = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$.

Since the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ are linearly independent (it is only two, and they are clearly not multiples of each other), these two vectors are a basis for V.

In particular, the dimension of V is 2.

It automatically follows from that (see the next theorem) that $V = \mathbb{R}^2$.

(b) Consider $W = \operatorname{row}\left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\right)$. By definition, $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$.

Note that $W = \operatorname{col}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$ and, in general, $\boxed{\operatorname{row}(A) = \operatorname{col}(A^T)}$.

[This means that every statement about column spaces translate into an equivalent statement for row spaces, and vice versa.]

Since the vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are linearly independent, and so are a basis for V.

Theorem 105. Let V be a vector space of dimension d.

- If the d vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ from V are independent, then they are a basis of V.
- If the d vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d$ span V, then they are a basis of V.

In particular, it follows from the first part that we can easily decide whether a bunch of vectors is a basis of $\mathbb{R}^n$: they are a basis if and only if these are exactly n vectors and they are independent.

Example 106. Which of the following sets are a basis for $\mathbb{R}^2$?

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Any basis of $\mathbb{R}^2$ has to have two vectors. So a single vector cannot constitute a basis for $\mathbb{R}^2$.

[Note this set of one vector is linearly independent (why?!) but $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \neq \mathbb{R}^2$.]

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

These are two vectors, and they are clearly independent. Hence, they are a basis of $\mathbb{R}^2$.

Recall that this special basis is called the standard basis.

(c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

These are two vectors, and they are clearly independent. Hence, they are a basis of $\mathbb{R}^2$.

Recall that this special basis is called the standard basis.

(d) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

These are two vectors, and they are clearly independent (why?!). Hence, they are a basis of $\mathbb{R}^2$.

[Note that, in particular, this implies that, without further computations, $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$.]

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(e) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

Any basis of $\mathbb{R}^2$ has to have two vectors. Three vectors cannot constitute a basis for $\mathbb{R}^2$.
[Note that we have already learned that three vectors in $\mathbb{R}^2$ cannot be independent.]

[Further note that $\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$, so these vectors span, but they are not a basis precisely because of the lack of independence. Independence means that there is no redundancy in the spanning vectors. Here, the third vector is redundant: $\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}$.]

Example 107. Which of the following sets are a basis for $\mathbb{R}^3$?

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Any basis of $\mathbb{R}^3$ has to have three vectors. Two vectors cannot constitute a basis for $\mathbb{R}^3$.

(b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

Any basis of $\mathbb{R}^3$ has to have three vectors. Four vectors cannot constitute a basis for $\mathbb{R}^3$.

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

These are three vectors, which is the right number for a basis of $\mathbb{R}^3$. They form a basis if and only if they are linearly independent (and we are experts in checking that).

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \underset{\rightsquigarrow}{\overset{R_2 - R_1 \Rightarrow R_2}{R_3 - R_1 \Rightarrow R_3}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

This (homogeneous) system (we are omitting the zero right-hand side) has no free variables. Hence, the three vectors are independent, and therefore a basis of $\mathbb{R}^3$.

(d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

These are three vectors, which is the right number for a basis of $\mathbb{R}^3$. They form a basis if and only if they are linearly independent (and we are experts in checking that).

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \underset{\rightsquigarrow}{\overset{R_2 - R_1 \Rightarrow R_2}{R_3 - R_1 \Rightarrow R_3}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \overset{R_3 - 2R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This (homogeneous) system (we are omitting the zero right-hand side) does have a free variable. Hence, the three vectors are dependent, and therefore not a basis of $\mathbb{R}^3$.

[By solving the system, we find the dependence relation $3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0}$.]

(e) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$ (final answer: not a basis)

(f) $\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 11 \\ 2 \end{bmatrix}$ (final answer: is a basis)