What we have learned so far

Linear systems

- Systems of equations can be written as $A\boldsymbol{x}=\boldsymbol{b}$.

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\boldsymbol{x}=\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

  Sometimes, we represent the system by its augmented matrix $\left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$.

- Thirdly, we can write the system in vector form as $x_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}+x_2\begin{bmatrix} -2 \\ 3 \end{bmatrix}=\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

  In this form, we see that one way to think about the system is that we are trying to write the right-hand side vector as a linear combination of two other vectors.

- A linear system has either
  - no solution (such a system is called **inconsistent**),

    $\Longleftrightarrow$ echelon form contains row $[\ 0\ \ \ldots\ \ 0 \,|\, b\ ]$ with $b\neq 0$
  - one unique solution,

    $\Longleftrightarrow$ system is consistent and has no free variables
  - infinitely many solutions.

    $\Longleftrightarrow$ system is consistent and has at least one free variable

- In order to solve the system $A\boldsymbol{x}=\boldsymbol{b}$ we do **Gaussian elimination** on $[\ A\ \ \boldsymbol{b}\ ]$.

  From the (unique!) RREF, we can read off the general solution. For instance:

$$\underbrace{\left[\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]}_{\text{RREF}} \Rightarrow \boldsymbol{x}=\begin{bmatrix} 4-2s_1+s_2 \\ s_1 \\ 3-s_2 \\ s_2 \end{bmatrix}=\underbrace{\begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix}}_{\text{particular solution}}+\underbrace{s_1\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}+s_2\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{general solution to homogeneous eq.}}.$$

Matrices and vectors

- A **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ is of the form

$$x_1\boldsymbol{v}_1+x_2\boldsymbol{v}_2+\ldots+x_m\boldsymbol{v}_m.$$

- $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is the set of all such linear combinations.

  For instance, a span in $\mathbb{R}^3$ can be $\{\boldsymbol{0}\}$, a line, a plane, or all of $\mathbb{R}^3$.

- To decide whether $\boldsymbol{w}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$,
- $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are **(linearly) independent** if there is only the trivial solution to

$$x_1\boldsymbol{v}_1+x_2\boldsymbol{v}_2+\ldots+x_n\boldsymbol{v}_n=\boldsymbol{0}.$$

Armin Straub
straub@southalabama.edu

Matrices

- An $m \times n$ **matrix** A has m rows and n columns.
- The **transpose** A^T of a matrix A has rows and columns flipped.

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ -1 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

- The product $A\boldsymbol{x}$ of **matrix times vector** is

$$\begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + \ldots + x_n \boldsymbol{a}_n.$$

- The **inverse** A^{-1} of A is characterized by $A^{-1}A = I$ (or $AA^{-1} = I$).
  - $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
  - Can compute A^{-1} using Gauss–Jordan method: $\begin{bmatrix} A & I \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} I & A^{-1} \end{bmatrix}$
- **Gaussian elimination** can bring any matrix into an **echelon form**.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And we can continue row reduction to obtain the (unique) **RREF**.

Using Gaussian elimination

Gaussian elimination and row reductions allow us to:

- solve systems of linear systems

$$\left[\begin{array}{cccc|c} 0 & 3 & -6 & 4 & -5 \\ 3 & -7 & 8 & 8 & 9 \\ 3 & -9 & 12 & 6 & 15 \end{array}\right] \overset{\text{RREF}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -24 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \rightsquigarrow \quad \boldsymbol{x} = \begin{bmatrix} -24 + 2s_1 \\ -7 + 2s_1 \\ s_1 \\ 4 \end{bmatrix}$$

- compute the inverse of a matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix} \quad \text{because} \quad \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

- determine whether a vector is a linear combination of other vectors

  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ if and only if the system $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 3 \end{array}\right]$ is consistent.

  (And each solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ gives a linear combination $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.)

Armin Straub
straub@southalabama.edu