Sketch of Lecture 11

Review. $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence, $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$.

Never write $\frac{A}{B}$ for matrices! Why?

Because it is unclear whether you mean AB^{-1} or $B^{-1}A$. (And order matters a lot with matrices!)

Example 82. Solve $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution. From the previous example we known that $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$.

We multiply $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ on both sides with $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$ to get

$$\underbrace{\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}}_{=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

which tells us that $\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

Theorem 83. If A is invertible, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Proof. Multiply both sides of $A\boldsymbol{x} = \boldsymbol{b}$ with A^{-1}. □

Example 84. The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible.

Why?!

- Reason 1: Because $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Reason 2: Because the system $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ clearly has no solution. (Compare with Theorem 83!)

The inverse of a 2×2 matrix

The following formula immediately gives us the inverse of a 2×2 matrix (if it exists). It is worth remembering!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ provided that $ad - bc \neq 0$

Let's check that! $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -cb+ad \end{bmatrix} = I_2$

Note 85.

- A 1×1 matrix $[\, a \,]$ is invertible $\Longleftrightarrow$ $a \neq 0$.
- A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Longleftrightarrow$ $ad - bc \neq 0$.

We will later see that the quantities on the RHS are the **determinants** of these matrices.

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Example 86. Solve $\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Solution. We multiply both sides with the inverse of $\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$.

[In the process of doing so, we see that the inverse does exist.]

Using the formula for 2×2 matrices:

$$\boldsymbol{x} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{14-15} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

Recipe for computing the inverse of any matrix

Theorem 83 tells us that, if A is invertible, then the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for any choice of $\boldsymbol{b}$ (namely, $\boldsymbol{x} = A^{-1}\boldsymbol{b}$). This property characterizes invertible matrices!

Theorem 87.

The linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for any choice of $\boldsymbol{b}$.

$\Longleftrightarrow$ The RREF of A is an identity matrix.

$\Longleftrightarrow$ A is invertible. (In particular, A has to be a square matrix!)

Why? If $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for any choice of $\boldsymbol{b}$, then the RREF of A cannot have a free variable, which means that every column of the RREF has to contain a pivot. Also, the RREF of A cannot have a zero row (because then the system will be inconsistent for certain $\boldsymbol{b}$), which means that every row of the RREF has to contain a pivot. But that means that the number of columns and rows has to be the same (both equal to the number of pivots) and, moreover that the RREF is all zeroes except 1's on the diagonal (the pivots). In other words, the RREF is an identity matrix.

To compute A^{-1}: **Gauss–Jordan method**

- Form the augmented matrix $[\ A \mid I\ ]$.
- Compute the reduced echelon form. (i.e. Gauss–Jordan elimination)
- If A is invertible, the RREF is of the form $[\ I \mid A^{-1}\ ]$.

Why is that reasonable?

- Well, to solve $A\boldsymbol{x} = \boldsymbol{b}$, we do row reduction on $[\ A \mid \boldsymbol{b}\ ]$.
- Likewise, to solve $AX = I$ (to find the inverse X), we do row reduction on $[\ A \mid I\ ]$.

Example 88. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, if it exists.

Solution. Let us do elimination on $[\ A \quad I_3\ ]$. If A is invertible, then the RREF will have the form $[\ I_3 \quad B\ ]$ and $A^{-1} = B$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \overset{R2 \to R2 - 2R1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Hence, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.