Example 70. Find the general solution of

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -4 & 2 & 4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

Solution. We eliminate:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ -2 & -4 & 2 & 4 & -2 \end{array}\right] \overset{R_2+2R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 0 & 0 & 2 & 2 & 6 \end{array}\right] \overset{\frac{1}{2}R_2 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

Our free variables are $x_2 = s_1$ and $x_4 = s_2$. We read off that $x_1 = 4 - 2s_1 + s_2$, $x_3 = 3 - s_2$. Hence, the general solution is

$$\boldsymbol{x} = \begin{bmatrix} 4 - 2s_1 + s_2 \\ s_1 \\ 3 - s_2 \\ s_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix}}_{\text{particular solution}} + \underbrace{s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{general solution to homogeneous eq.}}.$$

In accordance with Theorem 67, the solution is given as a particular solution $\boldsymbol{x}_p = [\,4\ \ 0\ \ 3\ \ 0\,]^T$ plus the general solution to the homogeneous equation $\begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -4 & 2 & 4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Linear independence

Example 71. The following is BAD! The general solution in the previous problem can also be written as

$$\boldsymbol{x} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix}.$$

[Check that $[\,0\ \ 1\ \ -2\ \ 2\,]^T$ indeed solves the homogeneous equation.]

Mathematically, this is correct. But it is morally bad!! Note how pretentious it is: it suggests that there are three free parameters in the solution when there is really only two.

The vector after s_3 is entirely redundant because

$$\begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Definition 72. Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are **(linearly) dependent** is there is a non-trivial linear combination

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_n\boldsymbol{v}_n = \boldsymbol{0}.$$

[There is always the trivial linear combination in which **all** coefficients are 0: $x_1 = 0$, $x_2 = 0$, …, $x_n = 0$.]

Otherwise, the vectors are **(linearly) independent**.

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Example 73. The vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix}$ are linearly dependent because

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a non-trivial linear combination of them which produces $\mathbf{0}$.

Example 74. Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ linearly independent?

Solution. We need to find out if

$$x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has any solutions besides the trivial solution $x_1 = x_2 = x_3 = 0$. But that's just asking whether a linear system (which is obviously consistent; why?!) has a unique solution or whether there are infinitely many solutions.

We therefore eliminate:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right] \underset{\rightsquigarrow}{\overset{R_2 - R_1 \Rightarrow R_2}{R_3 - R_1 \Rightarrow R_3}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \underset{\rightsquigarrow}{R_3 - 2R_2 \Rightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the echelon form, we see that the system is consistent (it had to be!) and that it has infinitely many solutions (because there is a free variable).

Hence, our three vectors are not linearly independent.

Example 75. Demonstrate that the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ are linearly dependent.

Solution. We have already done the bulk of the work in the previous problem.

For a change, let us solve the system by back-substitution. $x_3 = s_1$ is free. Then, $x_2 = -2s_1$ and $x_1 = -x_2 + x_3 = 3s_1$. This means that

$$3s_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2s_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s_1\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a non-trivial linear combination of our three vectors which produces the zero vector.

Note that setting s_1 produces a nice linear combination, and that every other linear combination is just a multiple.