The transpose of a matrix

Definition 62. Interchanging the rows and columns of A produces its **transpose** A^T.

Example 63.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}^T = \ldots$

(c) $\begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$

A matrix A such that $A^T = A$ is called **symmetric**.

(d) $[\, x_1 \;\; x_2 \;\; x_3 \,]^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

[This is useful for typographical reasons, because column vectors take up so much space.]

Theorem 64. Let A, B be matrices of appropriate size. Then:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$ (illustrated by the next example)

Example 65. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

Compute:

(a) $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} =$

(b) $(AB)^T = \begin{bmatrix} \quad \end{bmatrix}$

(c) $B^T A^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} =$

(d) $A^T B^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} =$ What's that fishy smell?

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Solving linear systems in the form $Ax = b$

Example 66. Our current homework asks if $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is in $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} \right\}$.

This question is the same (why?!) as asking whether the equation

$$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \qquad (1)$$

has a solution (i.e. whether the system is consistent).

As part of the homework, we eliminated to find

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \overset{\text{RREF}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

[Recall that the RREF is unique. If you found a different RREF, then a (likely, small) error must have occured.]

By Theorem 27, this system is consistent and so $[\, 2 \;\; -1 \;\; 6 \,]^T$ is in the given span.

Let us continue and solve (1). The RREF tells us that x_3 is a free variable, and we set $x_3 = s_1$. Then, $x_1 = 2 - 5s_1$ and $x_2 = 3 - 4s_1$. In vector notation, the general solution to (1) thus is

$$\boldsymbol{x} = \begin{bmatrix} 2 - 5s_1 \\ 3 - 4s_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}.$$

Theorem 67. Let $\boldsymbol{x}_p$ be a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$. Then all solutions of $A\boldsymbol{x} = \boldsymbol{b}$ have the form $\boldsymbol{x}_g = \boldsymbol{x}_p + \boldsymbol{x}_h$ where $\boldsymbol{x}_h$ is a solution to the associated **homogeneous system** $A\boldsymbol{x} = \boldsymbol{0}$.

Example 68. In the previous example, a particular solution is $\boldsymbol{x}_p = [\, 2 \;\; 3 \;\; 0 \,]^T$ (but any other solution would work just as well). On the other hand, $[\, -5 \;\; -4 \;\; 1 \,]^T$ solves the homogeneous system, as the following matrix-vector multiplication verifies:

$$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 69. Note that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a (particular) solution to $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Theorem 67 implies that the general solution is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}$.

[Because the associated homogeneous system is the same as before.]

Solve the system $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ from scratch and verify that your answer agrees!

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