Review 50.

(a) $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix} =$

(b) $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

Review 51. Note that this means that the system of equations

$$\begin{array}{rcrcr} x_1 & + & 4x_2 & = & 1 \\ 2x_1 & + & 5x_2 & = & -3 \\ 3x_1 & + & 6x_2 & = & 0 \end{array}$$

can be written compactly as

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}.$$

[This was the motivation for introducing matrix-vector multiplication.]

Likewise, any system can be written as $A\boldsymbol{x} = \boldsymbol{b}$, where A is a matrix and $\boldsymbol{b}$ a vector.

Why do we care?

- It's concise.
- The compactness sparks associations and ideas!
  - For instance, can we solve by *dividing* by A? $\boldsymbol{x} = A^{-1}\boldsymbol{b}$?
  - If $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{y} = 0$, then $A(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{b}$.
- Leads to matrix calculus and deeper understanding.
  - multiplying, inverting, or factoring matrices

Example 52. Suppose A is $m \times n$ and $\boldsymbol{x}$ is in $\mathbb{R}^p$. When does $A\boldsymbol{x}$ make sense?

Matrix times matrix

Example 53.

(a) $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 12 & -11 \end{bmatrix}$

because $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$

and $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -11 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 12 & -11 & 5 \end{bmatrix}$

Definition 54. Each column of AB is a linear combination of the columns of A with weights given by the corresponding column of B. For instance,

$$\begin{pmatrix} \text{col 5} \\ \text{of } AB \end{pmatrix} = A \begin{pmatrix} \text{col 5} \\ \text{of } B \end{pmatrix}$$

Example 55.

(a) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} =$

(b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} =$

Example 56.

(a) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

This is the 2×2 **identity matrix**.

Theorem 57. Let A, B, C be matrices of appropriate size. Then:

- $A(BC) = (AB)C$ associative
- $A(B + C) = AB + AC$ left-distributive
- $(A + B)C = AC + BC$ right-distributive

Example 58. However, matrix multiplication is not commutative!

(a) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix}$

Example 59. Also, a product can be zero even though none of the factors is:

$\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Example 60. Suppose A is $m \times n$ and B is $p \times q$.

(a) When does AB make sense? In that case, what are the dimensions of AB?

(b) When does BA make sense? In that case, what are the dimensions of BA?

Example 61. Consider the matrices

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}.$$

Compute (if possible) A_1B_1, A_2B_2 and B_2A_2. [Notice something?]

For more exercise, compute B_1B_2 and B_2B_1.