Working with spans

Review 41. $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is the set of all linear combinations

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_m\boldsymbol{v}_m,$$

where $x_1, x_2, \ldots, x_m$ can be any real numbers.

Example 42. Is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$?

If so, write $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Solution. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$ if and only if we can find x_1 and x_2 such that

$$x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

This is just the vector notation of the linear system with augmented matrix $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$.

This system is already in echelon form, and Theorem 27 tells us that it is consistent. Hence, our vector is in the given span.

Moreover, by back substitution, we find $x_2 = 1$ and $x_1 = 2$. This means

$$2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Example 43. Is $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right\}$? If so, write $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

Solution. As in the previous example, $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ is in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right\}$ if and only if the linear system $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 3 \end{array}\right]$ is consistent. We eliminate:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 3 \end{array}\right] \underset{\rightsquigarrow}{\overset{R_2 - R_1 \Rightarrow R_2}{R_3 - 2R_1 \Rightarrow R_3}} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{array}\right] \underset{\rightsquigarrow}{R_3 + 3R_2 \Rightarrow R_3} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

By Theorem 27, this system is consistent. Hence, our vector is in the given span.

By back substitution, we find $x_2 = -1$ and $x_1 = 2$. This means

$$2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}.$$

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Example 44. Is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right\}$? If so, write $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

Solution. The only thing that changes in the previous example, is the right-hand side of the linear system. This means that the elimination steps are exactly the same!

[This is important for applications. It is often the case that a system needs to be solved for many different right-hand sides.]

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 1 & 0 \end{array}\right] \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 - 2R_1 \Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -3 & -2 \end{array}\right] \overset{R_3 + 3R_2 \Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -5 \end{array}\right]$$

This system is inconsistent. Hence, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right\}$.

Matrix times vector

Definition 45. We say that A is a $m \times n$ matrix if it has m rows and n columns.

Example 46. $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ is a 2×3 matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ -1 & 1 & -3 \end{array}\right] \quad \begin{cases} \text{row picture} & \begin{array}{rcrcr} 2x_1 & + & 3x_2 & = & 1 \\ -x_1 & + & x_2 & = & -3 \end{array} \\ \text{column picture} & x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \end{cases}$$

We wish to write linear systems simply as $A\boldsymbol{x} = \boldsymbol{b}$. Here, $\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Example 47. For this, we need $\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Definition 48. The product of a matrix A with a vector $\boldsymbol{x}$ is a linear combination of the columns of A with weights given by the entries of $\boldsymbol{x}$. In other words,

$$A\boldsymbol{x} = x_1 \begin{pmatrix} \text{col } 1 \\ \text{of } A \end{pmatrix} + x_2 \begin{pmatrix} \text{col } 2 \\ \text{of } A \end{pmatrix} + \cdots$$

Example 49.

(a) $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 1 & 2 \\ 1 & -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} =$

(d) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} =$