The row-reduced echelon form

Review. Let us solve the following system by Gaussian elimination.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & - & 4x_2 & - & 5x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

First, we write down the **augmented matrix**, then we perform **elementary row operations** until we arrive at an equivalent **echelon form**:

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 3 & -4 & -5 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \overset{\substack{R_2-3R_1\Rightarrow R_2 \\ R_3+4R_1\Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \overset{R_3+\frac{3}{2}R_2\Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

(We can now stop and solve the system by back-substitution.) Instead we reduce further:

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{\substack{R_1-R_3\Rightarrow R_1 \\ R_2+8R_3\Rightarrow R_2}}{\rightsquigarrow} \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{\substack{R_1+R_2\Rightarrow R_1 \\ \frac{1}{2}R_2\Rightarrow R_2}}{\rightsquigarrow} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This is the **(row)-reduced echelon form**. From it, we can read off the solution $x_1 = 29$, $x_2 = 16$, $x_3 = 3$ without any further computations.

Definition 20. A matrix is in **(row)-reduced echelon form** (**RREF**) if, in addition to being in echelon form, it also satisfies:

(4) Each pivot is 1.

(5) Each pivot is the only nonzero entry in its column.

Example 21. The matrix from Example 18 put into reduced echelon form:

$$\left[\begin{array}{ccccccccccc} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccccccccccc} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Locate the **pivots**!

Armin Straub
straub@southalabama.edu

The general solution of a linear system

After row reduction to echelon form, we can easily solve a linear system.
(especially after reduction to reduced echelon form)

Example 22. Consider the following linear system:

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \rightsquigarrow \begin{array}{rrrrrl} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array}$$

- The augmented matrix is already in reduced echelon form. (Check!)
- The pivots are located in columns $1, 3, 5$.
  The corresponding variables x_1, x_3, x_5 are called **leading variables** (or **pivot variables**).
- The remaining variables x_2, x_4 are called **free variables**.
  We have no equations to solve for the free variables. Instead, the free variables can take any values.
  We set $x_2 = s_1$ and $x_4 = s_2$, where s_1, s_2 can be any numbers (free parameters).
- Solving each equation for the pivot variable, we find that the **general solution** (in parametric form) of this system is:

$$\begin{array}{rrrrl} x_1 & +6s_1 & & +3s_2 & = 0 \\ & & x_3 & -8s_2 & = 5 \\ & & & x_5 & = 7 \end{array} \qquad \begin{cases} x_1 = -6s_1 - 3s_2 \\ x_2 = s_1 \\ x_3 = 5 + 8s_2 \\ x_4 = s_2 \\ x_5 = 7 \end{cases}$$

Example 23. Find the general solution of the following linear system:

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & - & x_4 & = & 1 \\ 3x_1 & + & 6x_2 & - & 2x_3 & + & x_4 & = & 8 \end{array}$$

Solution. A single operation produces an echelon form: (Which are going to be our free variables?!)

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 1 \\ 3 & 6 & -2 & 1 & 8 \end{array}\right] \overset{R_2 - 3R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 1 \\ 0 & 0 & 1 & 4 & 5 \end{array}\right]$$

One more operation yields the reduced echelon form:

$$\overset{R_1 + R_2 \Rightarrow R_1}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 6 \\ 0 & 0 & 1 & 4 & 5 \end{array}\right] \qquad \begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & & & + & 3x_4 & = & 6 \\ & & & & x_3 & + & 4x_4 & = & 5 \end{array}$$

The free variables are x_2, x_4. We set $x_2 = s_1$ and $x_4 = s_2$. The general solution is:

$$\begin{aligned} x_1 &= 6 - 2s_1 - 3s_2 \\ x_2 &= s_1 \\ x_3 &= 5 - 4s_2 \\ x_4 &= s_2 \end{aligned}$$

[Here, s_1 and s_2 can be any numbers. The resulting values for x_1, x_2, x_3, x_4 always solve the system. Our solution is general, meaning that there are no further solutions.]