Sketch of Lecture 1

Tue, 8/18/2015

Introduction to systems of linear equations

Definition 1. A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written as

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

for some numbers $b, a_1, a_2, \ldots, a_n$.

Example 2. Which of the following equations are linear?

- $4x_1 - 5x_2 + 2 = x_1$ — linear: $3x_1 - 5x_2 = -2$
- $x_2 = 2(\sqrt{6} - x_1) + x_3$ — linear: $2x_1 + x_2 - x_3 = 2\sqrt{6}$
- $4x_1 - 6x_2 = x_1 x_2$ — not linear: $x_1 x_2$
- $x_2 = 2\sqrt{x_1} - 7$ — not linear: $\sqrt{x_1}$

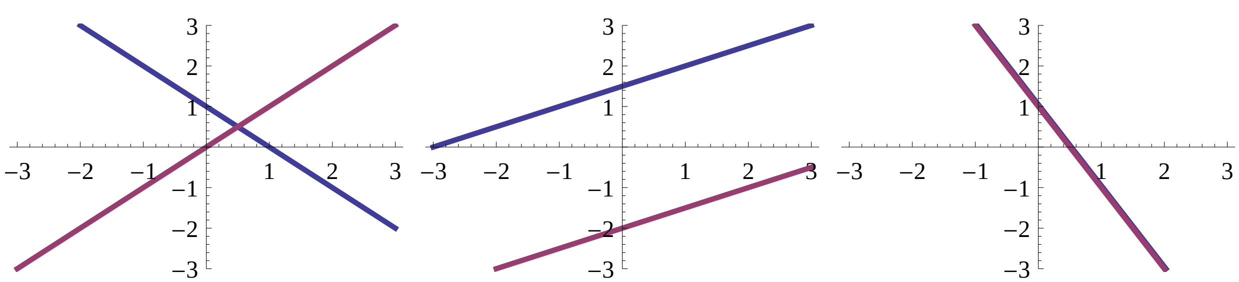
Definition 3.

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.
- A **solution** of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

Example 4. (Two equations in two variables)

In each case, sketch the set of all solutions.

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 1 \\ x_1 & - & x_2 & = & 0 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array} \qquad \begin{array}{rcrcr} 2x_1 & + & x_2 & = & 1 \\ -4x_1 & - & 2x_2 & = & -2 \end{array}$$



Theorem 5. A linear system has either

- no solution, or
- one unique solution, or
- infinitely many solutions.

Definition 6. A system is **consistent** if a solution exists.

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How to solve systems of linear equations

Strategy: replace system with an equivalent system which is easier to solve

Definition 7. Linear systems are **equivalent** if they have the same set of solutions.

Example 8. To solve the first system from the previous example:

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 1 \\ x_1 & - & x_2 & = & 0 \end{array} \quad \underset{\rightsquigarrow}{R_2 - R_1 \Rightarrow R_2} \quad \begin{array}{rcr} x_1 + x_2 & = & 1 \\ -2x_2 & = & -11 \end{array}$$

Once in this **triangular** form, we find the solutions by **back-substitution**:

$$x_2 = 1/2, \qquad x_1 = 1/2$$

Example 9. The same approach works for more complicated systems.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \quad \downarrow R_3 + 4R_1 \Rightarrow R_3$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \quad \downarrow R_3 + \tfrac{3}{2}R_2 \Rightarrow R_3$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & & & x_3 & = & 3 \end{array}$$

By back-substitution:

$$x_3 = 3, \qquad x_2 = 16, \qquad x_1 = 29.$$

It is always a good idea to check our answer. Let us check that $(29, 16, 3)$ indeed solves the original system:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{array}{rcrcrcr} 29 & - & 2\cdot 16 & + & 3 & \overset{\checkmark}{=} & 0 \\ & & 2\cdot 16 & - & 8\cdot 3 & \overset{\checkmark}{=} & 8 \\ -4\cdot 29 & + & 5\cdot 16 & + & 9\cdot 3 & \overset{\checkmark}{=} & -9 \end{array}$$

Example 10. Solve the following linear system:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 5 \end{array}$$

(To stick to our strategy, your first step should be $R_3 - 2R_1 \Rightarrow R_3$.)

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