Homework #8

MATH 237 — Linear Algebra I
due Thursday, Nov 5, in class

Please print your name:

Problem 1. For each of the following sets, decide whether they are a vector space. Briefly indicate your reasoning!

(a) $V = \left\{ \begin{bmatrix} a \\ 2 \\ 2a-b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

(b) $V = \left\{ \begin{bmatrix} a \\ 2b \\ 2a-b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

(c) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x - y = z \right\}$

(d) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x - y = 1 \right\}$

(e) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2 + y^2 + z^2 = 1 \right\}$

(f) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z \geqslant 0 \right\}$

Solution.

(a) V is not a vector space because it does not contain the zero vector.

(b) V is a vector space because $V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$. (Note that $\begin{bmatrix} a \\ 2b \\ 2a-b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.)

(c) V is a vector space because $V = \operatorname{null}(\begin{bmatrix} 2 & -1 & -1 \end{bmatrix})$.

(Note that $2x - y = z$ is equivalent to $\begin{bmatrix} 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$.)

(d) V is not a vector space because it does not contain the zero vector.

(e) V is not a vector space because it does not contain the zero vector.

(f) V is not a vector space because it contains $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ but not $-1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. □

Problem 2. Write $V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : 2x - y = w \right\}$ as a null space and determine a basis.

Armin Straub
straub@southalabama.edu

Solution. Note that $2x - y = w$ is equivalent to $\begin{bmatrix} 2 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0.$

Hence, $V = \text{null}(\begin{bmatrix} 2 & -1 & 0 & -1 \end{bmatrix})$.

The general solution to $\begin{bmatrix} 2 & -1 & 0 & -1 \end{bmatrix}\boldsymbol{x} = 0$ is given by

$$\boldsymbol{x} = \begin{bmatrix} \frac{1}{2}s_1 + \frac{1}{2}s_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = s_1\begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_3\begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

A basis for V is therefore given by $\begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. □

Armin Straub
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