Homework #7

MATH 237 — Linear Algebra I
due Thursday, Oct 29, in class

Please print your name:

Problem 1. Let $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \end{bmatrix}$.

(a) Find a basis for $\mathrm{col}(A)$, $\mathrm{row}(A)$ and $\mathrm{null}(A)$.

(b) Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ be your basis for $\mathrm{row}(A)$, and let $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s$ be your basis for $\mathrm{null}(A)$. Find a basis for $W = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s\}$. Conclude that $\dim W = 4$, and hence $W = \mathbb{R}^4$.

[*Comment:* What you observe here is always the case! Let A be a $m \times n$ matrix. Note that $\mathrm{row}(A)$ and $\mathrm{null}(A)$ are both subspaces of $\mathbb{R}^n$. Further, it follows from what we learned in class that $\dim \mathrm{row}(A) + \dim \mathrm{null}(A) = n$. What you observed here is that, in fact, taking the sum of $\mathrm{row}(A)$ and $\mathrm{null}(A)$ gives you all of $\mathbb{R}^n$.
"Not only the dimensions add up to the full dimension, but the spaces add up to the full space!"]

Solution.

(a) We compute the RREF of A (an echelon form is enough but, since we have to solve $A\boldsymbol{x} = \boldsymbol{0}$ to find a basis for $\mathrm{null}(A)$, a reduced echelon form comes in handy):

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \end{bmatrix} \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 - 3R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & -5 \\ 0 & 0 & 2 & 10 \end{bmatrix} \overset{\substack{R_3 + 2R_2 \Rightarrow R_3 \\ -R_2 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are the first and third. Hence, a basis for $\mathrm{col}(A)$ is: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$.

A basis for $\mathrm{row}(A)$ is given by the nonzero rows of the echelon form: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$.

The general solution to $A\boldsymbol{x} = \boldsymbol{0}$ is given by $x_2 = s_1$, $x_4 = s_2$ (our free variables) and $x_3 = -5s_2$, $x_1 = -2s_1 - 4s_2$. In vector form:

$$\boldsymbol{x} = \begin{bmatrix} -2s_1 - 4s_2 \\ s_1 \\ -5s_2 \\ s_2 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

Consequently, a basis for $\mathrm{null}(A)$ is given by: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$.

(b) $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$.

To find a basis for W, we eliminate:

$$\begin{bmatrix} 1 & 0 & -2 & -4 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 \\ 4 & 5 & 0 & 1 \end{bmatrix} \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_4 - 4R_1 \Rightarrow R_4}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 0 & 5 & 8 \\ 0 & 1 & 0 & -5 \\ 0 & 5 & 8 & 17 \end{bmatrix} \overset{\substack{\text{swap} \\ \text{rows}}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 5 & 8 & 17 \\ 0 & 0 & 5 & 8 \end{bmatrix}$$

$$\overset{R_3 - 5R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 8 & 42 \\ 0 & 0 & 5 & 8 \end{bmatrix} \overset{R_4 - \frac{5}{8}R_3 \Rightarrow R_4}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -2 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 8 & 42 \\ 0 & 0 & 0 & -\frac{73}{4} \end{bmatrix}$$

Armin Straub
straub@southalabama.edu

There is no free variables, so the four vectors in the span of W are independent.

Thus, a basis for W is: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$. Therefore, $\dim W = 4$ and so $W = \mathbb{R}^4$. □

Problem 2. Let A be a $m \times n$ matrix.

(a) For each of $\mathrm{col}(A)$, $\mathrm{row}(A)$ and $\mathrm{null}(A)$, state which space $\mathbb{R}^{??}$ they are a subspace of.

(b) Why is $\dim \mathrm{row}(A) + \dim \mathrm{null}(A) = n$?

(c) Suppose that the columns of A are independent. What can you say about the dimensions of $\mathrm{col}(A)$, $\mathrm{row}(A)$ and $\mathrm{null}(A)$?

(d) Suppose that A has **rank** 2 (that is, an echelon form of A has exactly 2 pivots). What can you say about the dimensions of $\mathrm{col}(A)$, $\mathrm{row}(A)$ and $\mathrm{null}(A)$?

Solution.

(a) $\mathrm{col}(A)$ is a subspace of $\mathbb{R}^m$.

$\mathrm{row}(A)$ is a subspace of $\mathbb{R}^n$.

$\mathrm{null}(A)$ is a subspace of $\mathbb{R}^n$.

(b) $\dim \mathrm{row}(A)$ is equal to the number of pivots, and $\dim \mathrm{null}(A)$ equal to the number of free variables.

There is n columns, and each corresponds to a pivot or a free variable. Hence, $\dim \mathrm{row}(A) + \dim \mathrm{null}(A) = n$.

(c) $\dim \mathrm{col}(A) = n$, $\dim \mathrm{row}(A) = n$ and $\dim \mathrm{null}(A) = 0$

(d) $\dim \mathrm{col}(A) = 2$, $\dim \mathrm{row}(A) = 2$ and $\dim \mathrm{null}(A) = n - 2$ (because A has n columns, 2 of which contain a pivot). □