Homework #5

MATH 237 — Linear Algebra I
due Tuesday, Sep 29, in class

Please print your name:

Problem 1. Consider $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Calculate A^{-1}.

(b) Using (a), solve the system $A\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

Solution.

(a)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_2 - R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_3 - R_2 \Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right]$$

$$\overset{\substack{R_2 + 2R_3 \Rightarrow R_2 \\ R_1 + R_3 \Rightarrow R_1 \\ -R_3 \Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

Hence, $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$.

(b) $\boldsymbol{x} = A^{-1}\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$ □

Problem 2. Consider the vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

(a) Are the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ linearly independent? (No computation needed!)

(b) Use part (b) of Problem 1 to write $\boldsymbol{v}_4$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

(c) Are the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ linearly independent? (Compare with part (a) of Problem 1!)
If no, then write down a non-trivial linear relation of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ which gives $\mathbf{0}$.

(d) Are the vectors $\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ linearly independent?
If no, then write down a non-trivial linear relation of $\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ which gives $\mathbf{0}$.

(e) Using your work in (c), decide whether the following statements are true or false:

The system $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution.	TRUE	FALSE
The system $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is always consistent.	TRUE	FALSE
The system $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ always has a unique solution.	TRUE	FALSE
The matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ is invertible.	TRUE	FALSE

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(f) Using your work in (d), decide whether the following statements are true or false:

The system $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution. TRUE FALSE

The system $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is always consistent. TRUE FALSE

The system $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ always has a unique solution. TRUE FALSE

The matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 1 & -1 \end{bmatrix}$ is invertible. TRUE FALSE

Solution.

(a) No, four vectors in $\mathbb{R}^3$ always are linearly dependent.

(b) To write $\boldsymbol{v}_4$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, means finding x_1, x_2, x_3, x_4 such that $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 = \boldsymbol{v}_4$. This is just the vector form of the system in part (b) of Problem 1, and we already know that it has the unique solution $x_1 = 0$, $x_2 = -3$, $x_3 = 2$. Here is the linear relation in all its glory:

$$0 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

(c) Note that the columns of the matrix A from Problem 1 are just our three vectors. These three vectors are linearly independent if and only if $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution. From Problem 1 we know that A is invertible, and so $A\boldsymbol{x} = \mathbf{0}$ has the unique solution $\boldsymbol{x} = A^{-1}\mathbf{0} = \mathbf{0}$. So, the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent.

(d) Note that the linear relation from part (b) does not really involve $\boldsymbol{v}_1$. Rearranging it, we have

$$-3 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \mathbf{0},$$

which means that the vectors $\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are linearly dependent.

[If you didn't notice that, you would have arrived at the same conclusion after the usual computation.]

(e) Note that this is just the matrix A from Problem 1. It is invertible, and any system $A\boldsymbol{x} = \boldsymbol{b}$ always has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. Hence, the answers are: TRUE, TRUE, TRUE, TRUE

(f) Note that the columns of the matrix are the vectors $\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$. These vectors are linearly dependent, hence the first answer is FALSE. As a consequence, the matrix does not have 3 pivots (it actually has 2 but that doesn't matter), which means that the corresponding RREF has a column without pivot (a free variable; so the third answer is FALSE) and a row without pivot (in the augmented matrix this a row with all zeros on the left, and so the system will be inconsistent for some choices of the right-hand side; hence the third answer is FALSE). Because the RREF has less than 3 pivots, the matrix is not invertible, so the fourth answer is FALSE again.

[Actually, our theorem from class tells us that, for square matrices, the four questions always have the same answer. So the only possibilities are TRUE, TRUE, TRUE, TRUE and FALSE, FALSE, FALSE, FALSE.]

□