Homework #4

MATH 237 — Linear Algebra I
due Thursday, Sep 17, in class

Please print your name:

Problem 1. Consider

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 3 & 2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}.$$

(a) Find (in vector form) the general solution to the linear system $A\boldsymbol{x} = \boldsymbol{b}$.

(b) From your answer in (a), deduce the general solution to the associated homogeneous linear system $A\boldsymbol{x} = \boldsymbol{0}$.

[You should not have to do any computations!]

(c) Verify that $\boldsymbol{x} = [\, 0 \;\; 1 \;\; 1 \;\; 1 \,]^T$ is a (particular) solution to the linear system $A\boldsymbol{x} = \boldsymbol{c}$.

[You should not solve the system. Just multiply a matrix with a vector.]

(d) Using the information from (b) and (c), find the general solution to the linear system $A\boldsymbol{x} = \boldsymbol{c}$.

[Again, you should not have to do any computations!]

(e) Compute $A^T A$. (Make sure that your answer is a symmetric 4×4 matrix.)

(f) **(Bonus)** Suppose that A is any $m \times n$ matrix. Can you give a reason why $A^T A$ is always a symmetric matrix?

Solution.

(a) Let's eliminate!

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 2 & 2 \\ 1 & -2 & 3 & 2 & 2 \end{array}\right] \overset{\substack{R_2+R_1 \Rightarrow R_2 \\ R_3-R_1 \Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \end{array}\right] \overset{R_3-R_2 \Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{\substack{\frac{1}{2}R_2 \Rightarrow R_2 \\ R_1-\frac{1}{2}R_2 \Rightarrow R_1}}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last step was optional but a RREF is a nice thing to have.

x_2 and x_4 are free variables, and we set $x_2 = s_1$, $x_4 = s_2$. Then, $x_1 = -1 + 2s_1 + s_2$ and $x_3 = 1 - s_2$. In vector form, the general solution therefore is

$$\boldsymbol{x} = \begin{bmatrix} -1 + 2s_1 + s_2 \\ s_1 \\ 1 - s_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

where s_1 and s_2 can be any real numbers.

(b) The general solution to $A\boldsymbol{x} = \boldsymbol{0}$ is

$$\boldsymbol{x} = s_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

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(c) We compute the matrix-vector product

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}.$$

(d) The general solution to $A\boldsymbol{x} = \boldsymbol{c}$ is

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

(e) We calculate

$$\begin{bmatrix} 1 & -1 & 1 \\ -2 & 2 & -2 \\ 1 & 1 & 3 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & 2 \\ 1 & -2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 3 & 0 \\ -6 & 12 & -6 & 0 \\ 3 & -6 & 11 & 8 \\ 0 & 0 & 8 & 8 \end{bmatrix},$$

which is indeed symmetric.

(f) Recall that a matrix B is symmetric if $B^T = B$.

Here, $B = A^T A$. In that case, $B^T = (A^T A)^T = A^T (A^T)^T = A^T A = B$ (do you see which rules we used here?). So, $B = A^T A$ is symmetric.

□