Homework #3

MATH 237 — Linear Algebra I
due Thursday, Sep 10, in class

Please print your name:

Problem 1. Determine if the vector $\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$.

Solution. Let's eliminate!

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right] \underset{\rightsquigarrow}{\substack{R_2+2R_1\Rightarrow R_2 \\ R_3-2R_1\Rightarrow R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{array}\right] \underset{\rightsquigarrow}{R_3-R_2\Rightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

This system is not consistent. Hence, $\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$.

[Note that we could see already after the first step that the system is not consistent! Do you see it?] □

Problem 2.

(a) Is $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}\right\}$?

(b) If possible, write $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$.

(c) Is there more than one way to write $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$?

Solution.

(a) Let's eliminate!

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \underset{\rightsquigarrow}{R_2+2R_1\Rightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right] \underset{\rightsquigarrow}{R_3-2R_2\Rightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent. Hence, $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}\right\}$.

(b) We solve the system (already in reduced echelon form!) to find $x_1 = 2-5s$, $x_2 = 3-4s$, $x_3 = s$, where s can be any value. This means that, for any choice of s,

$$\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = (2-5s)\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + (3-4s)\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + s\begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}. \qquad (1)$$

[For instance, $s = 0$ yields a particularly simple linear combination.]

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(c) There are infinitely many such ways. Any choice of s in (1) produces a different linear combination.

□

Problem 3. Calculate $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ -1 \end{bmatrix}$. [How does this relate to the previous problem?]

Solution. $$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ -1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

[This is the choice $s = -1$ in the previous problem.] □