# Homework #2

**MATH 237 — Linear Algebra I**
**due Thursday, Sep 3, in class**

***Please print your name:***

---

**Problem 1.** For what values of $h$ is the following system consistent?

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & h \\ x_1 & + & 2x_2 & = & 0 \\ x_1 & - & x_2 & = & 1 \end{array}$$

**Solution.** Let's eliminate!

$$\begin{bmatrix} 1 & 1 & h \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \underset{\rightsquigarrow}{\overset{R_2-R_1\Rightarrow R_2}{R_3-R_1\Rightarrow R_3}} \begin{bmatrix} 1 & 1 & h \\ 0 & 1 & -h \\ 0 & -2 & 1-h \end{bmatrix} \underset{\rightsquigarrow}{R_3+2R_2\Rightarrow R_3} \begin{bmatrix} 1 & 1 & h \\ 0 & 1 & -h \\ 0 & 0 & 1-3h \end{bmatrix}$$

Since the final matrix is in echelon form, the system is consistent if and only if $1-3h=0$. In other words, the system is consistent if and only if $h=1/3$.

[There was a typo in an earlier version of this homework, where the system was:

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & h \\ x_1 & + & 2x_1 & = & 0 \\ x_1 & - & x_2 & = & 1 \end{array}$$

Can you spot the typo? If we take the typo serious, then we eliminate instead

$$\begin{bmatrix} 1 & 1 & h \\ 3 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \underset{\rightsquigarrow}{\overset{R_2-3R_1\Rightarrow R_2}{R_3-R_1\Rightarrow R_3}} \begin{bmatrix} 1 & 1 & h \\ 0 & -3 & -3h \\ 0 & -2 & 1-h \end{bmatrix} \underset{\rightsquigarrow}{R_3-\frac{2}{3}R_2\Rightarrow R_3} \begin{bmatrix} 1 & 1 & h \\ 0 & -3 & -3h \\ 0 & 0 & 1+h \end{bmatrix},$$

and conclude that the system is consistent if and only if $h=-1$. (We can also avoid any elimination because the second equation forces $x_1=0$, which used in the third equation forces $x_2=-1$. Using these values in the first equation, we get $0-1=h$. Hence, all three equations can be satisfied if and only if $h=-1$.)] □

**Problem 2.** Consider the following system of linear equations:

$$\begin{array}{rrrrrcr} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = & -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = & 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = & 15 \end{array}$$

(a) Starting with the augmented matrix, perform Gaussian elimination (that is, apply elementary row operations to obtain an equivalent matrix in echelon form). (*Hint*: interchange rows first. Record all your row operations!)

(b) From the matrix in echelon form, decide whether this linear system is consistent. If it is consistent, does it have a unique solution or infinitely many?

(c) Further reduce the matrix in echelon form to row-reduced echelon form. (This is often called Gauss–Jordan elimination.) (As always, record all your row operations!)

(d) From the matrix in reduced echelon form, read off the general solution of the linear system.

Armin Straub
straub@southalabama.edu

**Solution.**

(a) Let's eliminate!

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \overset{R_1 \Leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \overset{R_2 - R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$
$$\overset{R_3 - \frac{3}{2}R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This matrix is in echelon form.

(b) The echelon form does not contain a row of the form $[\, 0 \;\; 0 \;\; \ldots \;\; 0 \,|\, b \,]$ with $b \neq 0$. This means that our linear system is consistent. Since $x_3$ and $x_4$ are free variables, the system has infinitely many solutions.

(c) We continue eliminating:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \overset{\substack{\frac{1}{3}R_1 \Rightarrow R_1 \\ \frac{1}{2}R_2 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \overset{\substack{R_1 - 2R_3 \Rightarrow R_1 \\ R_2 - R_3 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$
$$\overset{R_1 + 3R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This final matrix is in reduced echelon form.

(d) The free variables are $x_3, x_4$, and we set $x_3 = s_1$, $x_4 = s_2$, where $s_1, s_2$ can be any numbers. The general solution is:

$$\begin{cases} x_1 = -24 + 2s_1 - 3s_2 \\ x_2 = -7 + 2s_1 - 2s_2 \\ x_3 = s_1 \\ x_4 = s_2 \\ x_5 = 4 \end{cases}$$

□