# Preparing for the Final

**MATH 237 — Linear Algebra I**
**Thursday, Dec 10**

***Please print your name:***

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**Problem 1.** Go over all past quizzes!

**Problem 2.** Study the practice problems for the two midterm exams!

**Problem 3.** Retake the two midterm exams!

(A copy without solutions is available on our course website. Of course, you also find solutions there.)

## Additional problems covering the material since the second midterm

**Problem 4.** Find the eigenvalues and bases for the eigenspaces of $A$.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 1 & 6 \\ 2 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

**Solution.**

(a) By expanding by the first row twice, we find that the characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 4-\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2\begin{vmatrix} 4-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3(4-\lambda).$$

The eigenvalues are $\lambda = 1$ (with multiplicity 3) and $\lambda = 4$.

- For $\lambda = 4$, the eigenspace is null$\left(\begin{bmatrix} -3 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.
- For $\lambda = 1$, the eigenspace is null$\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$.

  [Do the row operations to find this basis of the null space! Although, here we can still "see" it.]

(b) By expanding by the second column, we find that the characteristic polynomial is

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 1-\lambda & 6 \\ 2 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & -2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(-\lambda(4-\lambda)+4) = (1-\lambda)(\lambda-2)^2.$$

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The eigenvalues are $\lambda = 1$ and $\lambda = 2$ (with multiplicity 2).

- For $\lambda = 1$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 0 & -2 \\ 1 & 0 & 6 \\ 2 & 0 & 3 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

  [If you don't see that, do the row operations to find the basis of the null space!]

- For $\lambda = 2$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -2 & 0 & -2 \\ 1 & -1 & 6 \\ 2 & 0 & 2 \end{bmatrix}\right)$.

  $$\begin{bmatrix} -2 & 0 & -2 \\ 1 & -1 & 6 \\ 2 & 0 & 2 \end{bmatrix} \overset{\substack{R_2 + \frac{1}{2}R_1 \Rightarrow R_2 \\ R_3 + R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} -2 & 0 & -2 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \overset{\substack{-\frac{1}{2}R_1 \Rightarrow R_1 \\ -R_2 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

  We see from here that $\operatorname{null}\left(\begin{bmatrix} -2 & 0 & -2 \\ 1 & -1 & 6 \\ 2 & 0 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$.

  [Note that is wasn't clear up front whether the eigenspace would have dimension 1 or dimension 2.]

(c) By expanding by the first column, we find that the characteristic polynomial is

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3.$$

The only eigenvalue is $\lambda = 0$ (with multiplicity 3).

- For $\lambda = 0$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(d) By expanding by the first column, we find that the characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = (1-\lambda)\lambda^2.$$

The eigenvalues are $\lambda = 0$ (with multiplicity 2) and $\lambda = 1$.

- For $\lambda = 1$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

- For $\lambda = 0$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

□

**Problem 5.** Compute the following determinants by expanding by the first column.

Here, $i$ is the imaginary unit. All you need to know about it is that $i^2 = -1$.

(a) $|\, 1 \,|$

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(b) $\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix}$

(e) $\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix}$

(f) Can you guess what the next determinant will be?

**Solution.**

$$\begin{vmatrix} 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} = 1 - i^2 = 2$$

$$\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} - i\begin{vmatrix} i & 0 \\ i & 1 \end{vmatrix} = 2 - i^2 = 3$$

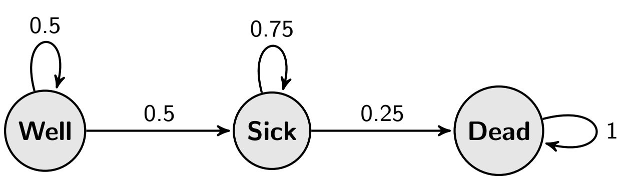
$$\begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} - i\begin{vmatrix} i & 0 & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 3 - i^2\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} = 5$$

$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

That's the Fibonacci numbers! The next determinant will be $5 + 8 = 13$. □

**Problem 6.** Suppose there is an epidemic in which, every month, half of those who are well become sick, and a quarter of those who are sick become dead. What is the proportion of dead people in the long term equilibrium (steady state).

**Solution.** The transition graph is:




$x_t$: proportion of those well at time $t$ (in months)

$y_t$: proportion of those sick at time $t$

$z_t$: proportion of those dead at time $t$

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 0.5x_t \\ 0.5x_t + 0.75y_t \\ 0.25y_t + z_t \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.75 & 0 \\ 0 & 0.25 & 1 \end{bmatrix}\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}$$

$\begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.75 & 0 \\ 0 & 0.25 & 1 \end{bmatrix}\begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix}$. That is, $\begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

Eigenspace of $\lambda=1$: $\text{null}\left(\begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ (do row operations if you don't see that right away!)

Since $x_\infty + y_\infty + z_\infty = 1$ in $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Hence, the proportion of dead people is $1 = 100\%$ in the long term equilibrium. □

## Some short answer problems

**Problem 7.** What are the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 4 \end{bmatrix}$?

**Solution.** The characteristic polynomial is $\det \begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ -1 & 3-\lambda & 0 & 0 \\ -1 & 1 & 3-\lambda & 0 \\ 0 & 1 & 2 & 4-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda)^2(4-\lambda)$.

Hence, the eigenvalues are 2, 3 (with multiplicity 2) and 4.

[We see from here that, for any triangular matrix, the eigenvalues are just its diagonal entries.] □

**Problem 8.** Write down the cofactor expansion of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ along

(a) the second row,

(b) the third column.

**Solution.**

(a) $\det(A) = -d \cdot \begin{bmatrix} b & c \\ h & i \end{bmatrix} + e \cdot \begin{bmatrix} a & c \\ g & i \end{bmatrix} - f \cdot \begin{bmatrix} a & b \\ g & h \end{bmatrix}$

(b) $\det(A) = c \cdot \begin{bmatrix} d & e \\ g & h \end{bmatrix} - f \cdot \begin{bmatrix} a & b \\ g & h \end{bmatrix} + i \cdot \begin{bmatrix} a & b \\ d & e \end{bmatrix}$ □

**Problem 9.** If $A$ and $B$ are $3 \times 3$ matrices with $\det(A) = 4$ and $\det(B) = -1$. What is the determinant of $C = 2A^T A^{-1} B A$?

**Solution.** We have

$$\det(C) = 2^3 \det(A^T)\det(A^{-1})\det(B)\det(A) = 8\det(A)\frac{1}{\det(A)}\det(B)\det(A) = 8\det(A)\det(B) = -32.$$

□

**Problem 10.** Let $A$ be a $7 \times 7$ matrix with $\dim \text{null}(A) = 1$. What can you say about $\det(A)$?

**Solution.** The null space of $A$ contains a nonzero vector. Hence, $A\boldsymbol{x} = \mathbf{0}$ has a nonzero solution, that is, $A$ is not invertible. Therefore, $\det(A) = 0$. □

**Problem 11.** Let $A$ be a $n \times n$ matrix with $A^T = A^{-1}$. What can you say about $\det(A)$?

**Solution.** If $A^T = A^{-1}$, then $\det(A^T) = \det(A^{-1})$ or, simplified, $\det(A) = \frac{1}{\det(A)}$. It follows that $(\det(A))^2 = 1$, which implies that $\det(A) = 1$ or $\det(A) = -1$. □

**Problem 12.** Let $A$ be a $5 \times 5$ matrix with $\dim \operatorname{row}(A) = 5$. What can you say about $\det(A)$?

**Solution.** Since $\dim \operatorname{row}(A) = 5$ and $A$ is $5 \times 5$, the matrix $A$ is invertible. Therefore, $\det(A) \neq 0$. □

**Problem 13.** What is $\dim \operatorname{null}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right)$?

**Solution.** The rank of that matrix is clearly 1. Hence, $\dim \operatorname{null}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right) = 3 - 1 = 2$. □

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