EXERCISES

In Exercises 1–6, perform the indicated computations when possible, using the matrices given below. If a computation is not possible, explain why.

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 4 & -5 \\ -2 & 1 & -3 \\ 0 & 2 & 6 \end{bmatrix}$$

1. (a) $A + B$, (b) $AB + I_2$, (c) $A + C$

2. (a) AC, (b) $C + D^T$, (c) $CB + I_2$

3. (a) $(AB)^T$, (b) CE, (c) $(A - B)D$

4. (a) A^3, (b) BC^T, (c) $EC + I_3$

5. (a) $(C + E)B$, (b) $B(C^T + D)$, (c) $E + CD$

6. (a) $AD - C^T$, (b) $AB - DC$, (c) $DE + CB$

In Exercises 7–10, find the missing values in the given matrix equation.

7. $\begin{bmatrix} 2 & a \\ 3 & -2 \end{bmatrix} \begin{bmatrix} b & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ 5 & c \end{bmatrix}$

8. $\begin{bmatrix} 1 & 4 \\ a & 7 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ b & 3 \end{bmatrix} = \begin{bmatrix} 6 & d \\ 11 & c \end{bmatrix}$

9. $\begin{bmatrix} a & 3 & -2 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & b \\ c & 1 \end{bmatrix} = \begin{bmatrix} 4 & d \\ -6 & -5 \end{bmatrix}$

10. $\begin{bmatrix} 1 & a \\ 0 & -2 \\ 5 & b \end{bmatrix} \begin{bmatrix} 3 & c & d \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 3 & 7 \\ e & -2 & -4 \\ f & -2 & 1 \end{bmatrix}$

11. Find all values of a such that $A^2 = A$ for

$$A = \begin{bmatrix} 5 & -10 \\ a & -4 \end{bmatrix}$$

12. Find all values of a such that $A^3 = 2A$ for

$$A = \begin{bmatrix} -2 & 2 \\ -1 & a \end{bmatrix}$$

13. Let T_1 and T_2 be linear transformations given by

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 5x_2 \\ -2x_1 + 7x_2 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + 9x_2 \\ 5x_2 \end{bmatrix}$$

Find the matrix A such that

(a) $T_1(T_2(\mathbf{x})) = A\mathbf{x}$

(b) $T_2(T_1(\mathbf{x})) = A\mathbf{x}$

(c) $T_1(T_1(\mathbf{x})) = A\mathbf{x}$

(d) $T_2(T_2(\mathbf{x})) = A\mathbf{x}$

14. Let T_1 and T_2 be linear transformations given by

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + 3x_2 \\ x_1 + 6x_2 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 5x_2 \\ x_1 + 5x_2 \end{bmatrix}$$

Find the matrix A such that

(a) $T_1(T_2(\mathbf{x})) = A\mathbf{x}$

(b) $T_2(T_1(\mathbf{x})) = A\mathbf{x}$

(c) $T_1(T_1(\mathbf{x})) = A\mathbf{x}$

(d) $T_2(T_2(\mathbf{x})) = A\mathbf{x}$

In Exercises 15–18, expand each of the given matrix expressions and combine as many terms as possible. Assume that all matrices are $n \times n$.

15. $(A + I)(A - I)$

16. $(A + I)(A^2 + A)$

17. $(A + B^2)(BA - A)$

18. $A(A + B) + B(B - A)$

In Exercises 19–22, the given matrix equation is *not* true in general. Explain why. Assume that all matrices are $n \times n$.

19. $(A + B)^2 = A^2 + 2AB + B^2$

20. $(A - B)^2 = A^2 - 2AB + B^2$

21. $A^2 - B^2 = (A - B)(A + B)$

22. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

23. Suppose that A has four rows and B has five columns. If AB is defined, what are its dimensions?

24. Suppose that A has four rows and B has five columns. If BA is defined, what are its dimensions?

In Exercises 25–28,

$$A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 0 & 1 & 4 \\ -1 & 2 & -2 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -1 & 1 \\ -3 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \\ 2 & 2 & -1 & -2 \end{bmatrix}$$

25. Partition A and B into four 2×2 blocks, and then use them to compute each of the following:

(a) $A - B$

(b) AB

(c) BA

26. Partition A and B into four blocks, with the upper left of each a 3×3 matrix, and then use them to compute each of the following:

(a) $A + B$

(b) AB

(c) BA

27. Partition A and B into four blocks, with the lower left of each a 3×3 matrix, and then use them to compute each of the following:

(a) $B - A$

(b) AB

(c) $BA + A$

28. Partition A and B into four blocks, with the lower right of each a 3×3 matrix, and then use them to compute each of the following:

(a) $A + B$

(b) AB

(c) BA

29. Suppose that A is a 3×3 matrix. Find a 3×3 matrix E such that the product EA is equal to A with

(a) the first and second rows interchanged.

(b) the first and third rows interchanged.

(c) the second row multiplied by -2.

30. Suppose that A is a 4×3 matrix. Find a 4×4 matrix E such that the product EA is equal to A with

(a) the first and fourth rows interchanged.

(b) the second and third rows interchanged.

(c) the third row multiplied by -2.

FIND AN EXAMPLE For Exercises 31–38, find an example that meets the given specifications.

31. 3×3 matrices A and B such that $AB \neq BA$.

32. 3×3 matrices A and B such that $AB = BA$.

33. 2×2 nonzero matrices A and B (other than those given earlier) such that $AB = 0_{22}$.

34. 3×3 nonzero matrices A and B such that $AB = 0_{33}$.

35. 2×2 matrices A and B (other than those given earlier) that have *no* zero entries and yet $AB = 0_{22}$.

36. 3×3 matrices A and B that have *no* zero entries and yet $AB = 0_{33}$.

37. 2×2 matrices A, B, and C (other than those given earlier) that are nonzero, where $A \neq B$ but $AC = BC$.

38. 3×3 matrices A, B, and C that are nonzero, where $A \neq B$ but $AC = BC$.

TRUE OR FALSE For Exercises 39–48, determine if the statement is true or false, and justify your answer. You may assume that A, B, and C are $n \times n$ matrices.

39. If A and B are nonzero (that is, not equal to 0_{nn}), then so is $A + B$.

40. If A and B are diagonal matrices, then so is $A - B$.

41. If A is upper triangular, then A^T is lower triangular.

42. $AB \neq BA$

43. $C + I_n = C$

44. If A is symmetric, then so is $A + I_n$.

45. $(ABC)^T = C^T B^T A^T$

46. If $AB = BA$, then either $A = I_n$ or $B = I_n$.

47. $(AB + C)^T = C^T + B^T A^T$

48. $(AB)^2 = A^2 B^2$

49. Prove the remaining unproven parts of Theorem 3.11.

(a) $A + B = B + A$

(b) $s(A + B) = sA + sB$

(c) $(s + t)A = sA + tA$

(d) $(A + B) + C = A + (B + C)$

(f) $A + 0_{nm} = A$

50. Prove the remaining unproven parts of Theorem 3.13.

(a) $A(BC) = (AB)C$

(b) $A(B + C) = AB + AC$

(d) $s(AB) = (sA)B = A(sB)$

(f) $IA = A$

51. Prove the remaining unproven parts of Theorem 3.15.

(a) $(A + B)^T = A^T + B^T$

(b) $(sA)^T = sA^T$

52. Verify Equation (2): If A is an $n \times m$ matrix and I_n is the $n \times n$ identity matrix, then $A = I_n A$.

53. Show that if A and B are symmetric matrices and $AB = BA$, then AB is also a symmetric matrix.

54. Let A and D be $n \times n$ matrices, and suppose that the only nonzero terms of D are along the diagonal. Must $AD = DA$? If so, prove it. If not, give a counter-example.

55. Let A be an $n \times m$ matrix.

(a) What are the dimensions of $A^T A$?

(b) Show that $A^T A$ is symmetric.

56. Suppose that A and B are both $n \times n$ diagonal matrices. Prove that AB is also an $n \times n$ diagonal matrix. (HINT: The formula given in (1) can be helpful here.)

57. Suppose that A and B are both $n \times n$ upper triangular matrices. Prove that AB is also an $n \times n$ upper triangular matrix. (HINT: The formula given in (1) can be helpful here.)

58. Suppose that A and B are both $n \times n$ lower triangular matrices. Prove that AB is also an $n \times n$ lower triangular matrix. (HINT: The formula given in (1) can be helpful here.)

59. Prove Theorem 3.17: If A is an upper (lower) triangular matrix and $k \geq 1$ is an integer, then A^k is also an upper (lower) triangular matrix.

60. If A is a square matrix, show that $A + A^T$ is symmetric.

61. A square matrix A is **skew symmetric** if $A^T = -A$.

(a) Find a 3×3 skew symmetric matrix.

(b) Show that the same numbers must be on the diagonal of all skew symmetric matrices.

62. A square matrix A is **idempotent** if $A^2 = A$.

(a) Find a 2×2 matrix, not equal to 0_{22} or I, that is idempotent.

(b) Show that if A is idempotent, then so is $I - A$.

63. If A is a square matrix, show that $(A^T)^T = A$.

64. The **trace** of a square matrix A is the sum of the diagonal terms of A and is denoted by $\text{tr}(A)$.

(a) Find a 3×3 matrix A with nonzero entries such that $\text{tr}(A) = 0$.

(b) If A and B are both $n \times n$ matrices, show that $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(A)$.

(c) Show that $\text{tr}(A) = \text{tr}(A^T)$.

(d) Select two nonzero 2×2 matrices A and B of your choosing, and check if $\text{tr}(AB) = \text{tr}(A)\text{tr}(B)$.

65. C In Example 7, suppose that the current distribution is 8000 homes with cable, 1500 homes with satellite, and 500 homes with no TV. Find the distribution one year, two years, three years, and four years from now.

66. C In Example 7, suppose that the current distribution is 5000 homes with cable, 3000 homes with satellite, and 2000 homes with no TV. Find the distribution one year, two years, three years, and four years from now.

67. C In an office complex of 1000 employees, on any given day some are at work and the rest are absent. It is known that if an employee is at work today, there is an 85% chance that she will be at work tomorrow, and if the employee is absent today, there is a 60% chance that she will be absent tomorrow. Suppose that today there are 760 employees at work. Predict the number that will be at work tomorrow, the following day, and the day after that.

68. C The star quarterback of a university football team has decided to return for one more season. He tells one person, who in turn tells someone else, and so on, with each person talking to someone who has not heard the news. At each step in this chain, if the message heard is "yes" (he is returning) then there is a 10% chance it will be changed to "no," and if the message heard is "no," then there is a 15% chance that it will be changed to "yes." Determine the probability that the fourth person in the chain hears the correct news.

C In Exercises 69–74, perform the indicated computations when possible, using the matrices given below. If a computation is not possible, explain why.

$$A = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & 3 & 3 & -1 \\ 6 & 8 & 1 & 1 \\ 5 & -3 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -6 & 2 & -3 & 1 \\ -5 & 2 & 0 & 3 \\ 0 & 3 & -1 & 4 \\ 8 & 5 & -2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 5 & 1 & 2 & 4 & 3 \\ 6 & 2 & 4 & 0 & 8 \\ 7 & 3 & 3 & 3 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 1 & 3 \\ 0 & 7 & 1 & 4 \\ 3 & 6 & 9 & 2 \\ 1 & 4 & 7 & 1 \end{bmatrix}$$

69. (a) $A + B$, (b) $BA - I_4$, (c) $D + C$

70. (a) AC, (b) $C^T - D^T$, (c) $CB + I_2$

71. (a) AB, (b) CD, (c) $(A - B)C^T$

72. (a) B^4, (b) BC^T, (c) $D + I_4$

73. (a) $(C + A)B$, (b) $C(C^T + D)$, (c) $A + CD$

74. (a) $AB - D^T$, (b) $AB - DC$, (c) $D + CB$