Diagonalization

Diagonal matrices are very easy to work with.

Example 1. For instance, it is easy to compute their powers.

If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then $A^{100} =$

Example 2. If $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$, then $A^{100} = ?$

Solution.

The key idea of the previous example was to work with respect to a basis given by the eigenvectors.

- Put the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ as columns into a matrix P.

$$A\boldsymbol{x}_i = \lambda\boldsymbol{x}_i \quad \Longrightarrow \quad A\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ | & & | \end{bmatrix}$$

- In summary: $AP = PD$

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Suppose that A is $n \times n$ and has independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.

Then A can be **diagonalized** as $A = PDP^{-1}$.

- the columns of P are
- the diagonal matrix D has

Such a diagonalization is possible if and only if A has enough eigenvectors.

Example 3. Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Solution.

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Definition 4. Matrices A and B are **similar** if there is an invertible matrix P such that

$$A = PBP^{-1}$$

Note that, in that case, $B = P^{-1}AP$. So the definition works both ways.

Example 5. So, another way to say that a matrix A can be diagonalized as $A = PDP^{-1}$ is that A is similar to a diagonal matrix.

In that case, what is A^n?

Solution.

Theorem 6. Similar matrices have the same characteristic polynomial (and hence the same eigenvalues).

Proof. Suppose that $A = PBP^{-1}$.

□

Practice problems

Problem 1. Find, if possible, the diagonalization of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

Problem 2. Find, if possible, the diagonalization of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

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