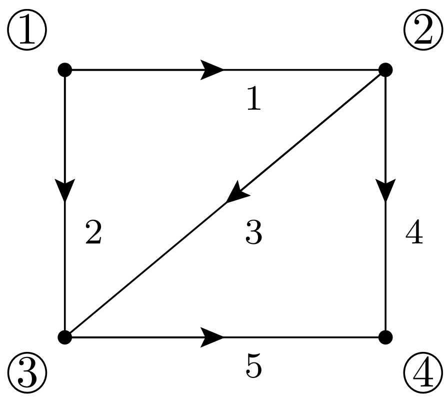
Application: directed graphs



- Graphs appear in network analysis (e.g. internet) or circuit analysis.
- arrow indicates direction of flow
- no edges from a node to itself
- at most one edge between nodes

Definition 1. Let G be a graph with m edges and n nodes.
The **edge-node incident matrix** of G is the $m \times n$ matrix A with

$$A_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j, \\ +1, & \text{if edge } i \text{ enters node } j, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2. Give the edge-node incidence matrix of our graph.

Solution.

Meaning of the null space

The $\boldsymbol{x}$ in $A\boldsymbol{x}$ is assigning values to each node.

You may think of assigning **potentials** to each node.

$$\mathbf{0} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =$$

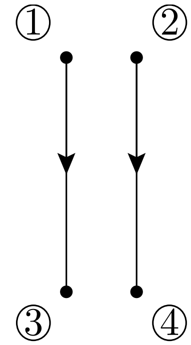
So: $A\boldsymbol{x} = \mathbf{0} \iff$

Armin Straub
astraub@illinois.edu

For our graph: $\text{Nul}(A)$ is spanned by ...

This always happens as long as the graph is **connected**.

Example 3. Give a basis for $\text{Nul}(A)$ for the following graph.



Solution.

In general:
$\dim \text{Nul}(A)$ is

For large graphs, disconnection may not be apparent visually.
But we can always find out by computing $\dim \text{Nul}(A)$ using Gaussian elimination!

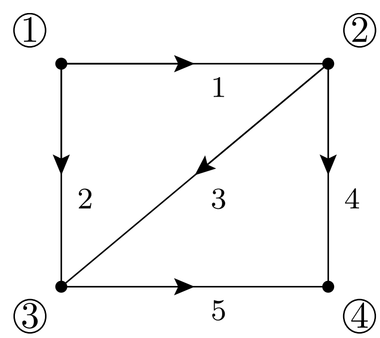
Meaning of the left null space

The $\boldsymbol{y}$ in $\boldsymbol{y}^T A$ is assigning values to each edge.

You may think of assigning **currents** to each edge.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} =$$



So: $A^T \boldsymbol{y} = \boldsymbol{0} \iff$

This is **Kirchhoff's first law**.

What is the simplest way to balance current?

Armin Straub
astraub@illinois.edu

Example 4. Suppose we did not "see" this.

Let us solve $A^T\boldsymbol{y} = \boldsymbol{0}$ for our graph:

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

In general:

$\dim \operatorname{Nul}(A^T)$ is

Meaning of the column space

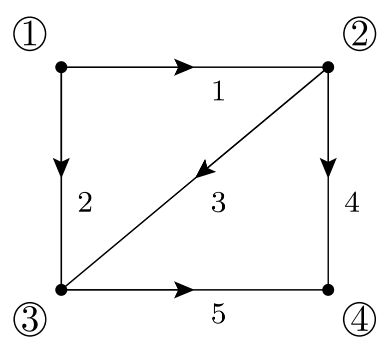
- FTLA: $\boldsymbol{b}$ is in $\operatorname{Col}(A)$ $\iff$ $\boldsymbol{b}$ is orthogonal to $\operatorname{Nul}(A^T)$
- Just found: $\operatorname{Nul}(A^T)$ has basis

Hence, $\boldsymbol{b}$ is in $\operatorname{Col}(A)$ if and only if

$\boldsymbol{x}$ assigns potentials to each node.

Then: $A\boldsymbol{x}$ are potential differences.

$A\boldsymbol{x} = \boldsymbol{b}$ is solvable if the potential differences $\boldsymbol{b}$ satisfy the constraints coming from $\operatorname{Nul}(A^T)$.



So: $\boldsymbol{b}$ is in $\operatorname{Col}(A)$ $\iff$

This is **Kirchhoff's second law**.

Meaning of the row space

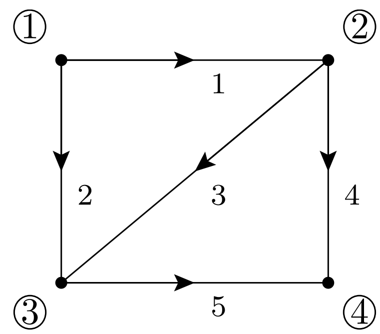
- FTLA: $\boldsymbol{f}$ is in $\mathrm{Col}(A^T)$ $\iff$ $\boldsymbol{f}$ is orthogonal to $\mathrm{Nul}(A)$
- Just found: $\mathrm{Nul}(A)$ has basis

Hence, $\boldsymbol{f}$ is in $\mathrm{Col}(A^T)$ if and only if

$\boldsymbol{y}$ assigns currents to each edge.

Then: $A^T\boldsymbol{y}$ are the net currents at each node.

$A^T\boldsymbol{y} = \boldsymbol{f}$ is solvable if the net currents $\boldsymbol{f}$ satisfy the constraints coming from $\mathrm{Nul}(A)$.



So: $\boldsymbol{f}$ is in $\mathrm{Col}(A^T)$ $\iff$

- Recall: linear dependencies among rows $\leftrightsquigarrow$ solutions to $\boldsymbol{y}^T A = 0$
- $\mathrm{Nul}(A^T)$ has basis
- A subset of the rows is independent $\iff$
- A subset of the rows is a basis for $\mathrm{Col}(A^T)$

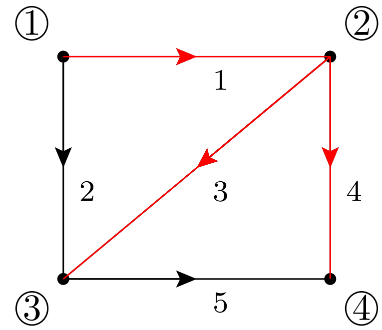
  $\iff$

Armin Straub
astraub@illinois.edu

A spanning tree:

- includes all nodes ("spans"),
- does not contain loops ("tree").

The choice to the right corresponds to



for basis of the row space of

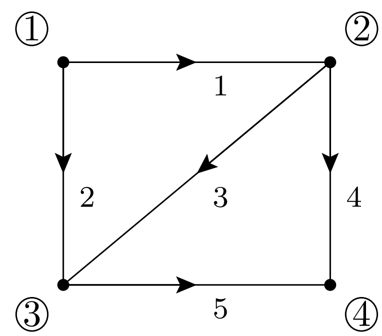
$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Euler's formula

Let G be a connected graph.

$$\#\text{nodes} - \#\text{edges} + \#\text{loops} = 1$$

Here:



Proof. Let A be the $m \times n$ edge-node incidence matrix of G.

□

Armin Straub
astraub@illinois.edu